Independence

<u>Def</u>: A probability space is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\Omega)=1$.

Elements of $\mathcal{F}$ are called events; elements of Ω are called elementary events.

Events E_1, E_2 are independent if $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2)$

Events $(E_i, i \in I)$ are (mutually) indep if for all $J \subset I$ finite, $\mathbb{P}(\bigcap_{j \in J} E_j) = \prod_{j \in J} \mathbb{P}(E_j)$

k-wise indep. if for all $J \subset I$, $|J| \leq k$, ...

Exercise: Convince yourself that k-wise indep. $\not\Rightarrow$ mutual indep.

Example (One easy trick to model an infinite sequence of fair coin tosses)

$\Omega = [0,1]$; $\mathcal{F} = \mathcal{B}([0,1]) := \mathcal{B}(\mathbb{R})|_{[0,1]}$. $\mathbb{P}$ = Lebesgue measure on $[0,1]$.

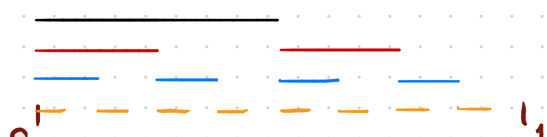
Events:

$A_1 = (0, \frac{1}{2}]$

$A_2 = (0, \frac{1}{4}] \cup (\frac{1}{2}, \frac{3}{4}]$

$A_3 = (0, \frac{1}{8}] \cup (\frac{1}{4}, \frac{3}{8}] \cup (\frac{1}{2}, \frac{5}{8}] \cup (\frac{3}{4}, \frac{7}{8}]$

$$A_k = \bigcup_{\substack{0 \leq i < 2^k \\ i \text{ even}}} \left(\frac{i}{2^k}, \frac{i+1}{2^k}\right]$$



Then $(A_i, i \geq 1)$ are indep.

<u>NB</u> $A_i = \{x \in (0,1] : \text{the } i\text{'th bit in the binary expansion of } x \text{ is } 0\}$

Convention: no ∞ strings of 0's.

Borel-Cantelli Lemmas

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E_n, n \geq 1)$ be events.

We define $\limsup_{n\to\infty} E_n := \bigcap_{n\geq 1} \bigcup_{m \geq n} E_m = \{\omega : \omega \in E_n \text{ for } \infty \text{ many } n\} =: \{E_n \text{ i.o.}\}$

$\liminf_{n\to\infty} E_n := \bigcup_{n\geq 1} \bigcap_{m \geq n} E_m = \{\omega : \omega \in E_n \text{ for all but finitely many } n\}$

Example: $\limsup_{n\to\infty} A_n = \{x \in [0,1] :$ There are infinitely many "0"s in any binary expansion of $x\}$

$\liminf_{n\to\infty} A_n = \{x \in [0,1] : x \text{ is a dyadic rational}\} = \{x : x = \frac{k}{2^n}, \text{some } n \geq 1, 0 \leq k \leq 2^n\}$

Exercise $(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty} (E_n^c)$

<u>NB</u> $\limsup E_n$, $\liminf E_n$ again in $\mathcal{F}$

First Borel-Cantelli Lemma

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E_n, n \geq 1)$ be events in $\mathcal{F}$.

If $\sum_{n\geq 1} \mathbb{P}(E_n) < \infty$ then $\mathbb{P}(E_n \text{ i.o.}) = 0$.

Proof Fix $\varepsilon > 0$. Then $\exists\, n_0$ s.t. $\sum_{m \geq n_0} \mathbb{P}(E_m) < \varepsilon$, so

$$\mathbb{P}(E_n \text{ i.o.}) \leq \mathbb{P}\Big(\bigcap_{n\geq n_0} \bigcup_{m\geq n} E_m\Big) \leq \mathbb{P}\Big(\bigcup_{m \geq n_0} E_m\Big) \leq \sum_{m\geq n_0} \mathbb{P}(E_m) < \varepsilon \qquad \square$$

Second Borel-Cantelli Lemma

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E_n, n \geq 1)$ be events. If $\sum_{n \geq 1} \mathbb{P}(E_n) = \infty$ and $(E_n, n \geq 1)$ are mutually independent then $\mathbb{P}(E_n \text{ i.o.}) = 1$.

Proof We'll prove $\mathbb{P}(\{E_n \text{ i.o.}\}^c) = 0$. Note: $\{E_n \text{ i.o.}\}^c = \liminf_{n \to \infty} (E_n^c) = \bigcup_{n \geq 1} \bigcap_{m \geq n} E_m^c$

Write $p_n := \mathbb{P}(E_n)$. Then for all $n \geq 1$ and all $N \geq n$,

$$\mathbb{P}\Big(\bigcap_{m \geq n} E_m^c\Big) \leq \mathbb{P}\Big(\bigcap_{m=n}^{N} E_m^c\Big) = \prod_{m=n}^{N} (1 - p_n) \underset{1-x \leq e^{-x}}{\leq} \exp\Big(-\sum_{m=n}^{N} p_n\Big) \xrightarrow[N \to \infty]{} 0$$

so LHS $= 0$ and thus

$$\mathbb{P}\Big(\bigcup_{n \geq 1} \bigcap_{m \geq n} E_m^c\Big) \leq \sum_{n \geq 1} \mathbb{P}\Big(\bigcap_{m \geq n} E_m^c\Big) = \sum_{n \geq 1} 0 = 0. \qquad \square$$

Definition Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space, let $(\mathcal{G}_i, i \in I)$ be sub-σ-fields of $\mathcal{F}$. Say $(\mathcal{G}_i, i \in I)$ are *independent* if for any collection $(E_i, i \in I)$ with $E_i \in \mathcal{G}_i \ \forall i \in I$, $(E_i, i \in I)$ are mutually indep.

Exercises

- Events $(E_i, i \geq 1)$ mutually ind. $\iff$ σ-fields $(\{\emptyset, E_i, E_i^c, \Omega\}, i \geq 1)$ indep.
- If $(E_i, i \in I)$ mutually ind and $(I_n, n \geq 1)$ partitions I, then $(\sigma(E_i, i \in I_n), n \geq 1)$ are ind σ-fields.

Theorem (A time-saving device for proving independence of σ-fields.)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{P}, \mathcal{Q} \subset \mathcal{F}$ be π-systems.
If $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{P}$, $B \in \mathcal{Q}$ then $\sigma(\mathcal{P})$ and $\sigma(\mathcal{Q})$ are independent.

Proof

Fix $A \in \mathcal{P}$, define measures μ_A, $\mathbb{P}_A$ on $\sigma(\mathcal{Q})$ by $\mu_A(B) := \mathbb{P}(A)\mathbb{P}(B)$, $\mathbb{P}_A(B) := \mathbb{P}(A \cap B)$.
Then $\mu_A \equiv \mathbb{P}_A$ on $\mathcal{Q}$, and $\mu_A(\Omega) = \mathbb{P}(A) = \mathbb{P}_A(\Omega)$, so $\mu_A = \mathbb{P}_A$ by Dynkin's Theorem.
Thus $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{P}$, $B \in \sigma(\mathcal{Q})$.

Next fix $B \in \sigma(\mathcal{Q})$, define measures ν_B, $\mathbb{P}_B$ on $\sigma(\mathcal{P})$ by $\nu_B(A) := \mathbb{P}(A)\mathbb{P}(B)$, $\mathbb{P}_B(A) := \mathbb{P}(A \cap B)$.
Then $\nu_B \equiv \mathbb{P}_B$ on $\mathcal{P}$, and $\nu_B(\Omega) = \mathbb{P}(B) = \mathbb{P}_B(\Omega)$, so $\nu_B = \mathbb{P}_B$ by Dynkin's theorem.
Thus $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \sigma(\mathcal{P})$, $B \in \sigma(\mathcal{Q})$, i.e. $\sigma(\mathcal{P})$ and $\sigma(\mathcal{Q})$ are indep. $\square$

Exercise If $(\mathcal{P}_i, i \in I)$ are π-systems $\subset \mathcal{F}$ and $\forall J \subset I$ finite, $\forall$ events $E_j \in \mathcal{P}_j$, $j \in J$, $\mathbb{P}\left(\bigcap_{j \in J} E_j\right) = \prod_{j \in J} \mathbb{P}(E_j)$, then $(\sigma(\mathcal{P}_i), i \in I)$ are indep. σ-algebras.

Random Variables

Def Given measurable spaces $(R, \mathcal{R})$ and $(S, \mathcal{S})$, an $(\mathcal{R}/\mathcal{S})$-measurable map is a function $X: R \to S$ with $X^{-1}(U) \in \mathcal{R}$ for all $U \in \mathcal{S}$.

A real-valued random variable is an $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable map $X: \Omega \to \mathbb{R}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Important notation $\{X \in U\} := X^{-1}(U)$, so $\mathbb{P}(X \in U)$ means $\mathbb{P}(X^{-1}(U)) = \mathbb{P}(\{\omega : X(\omega) \in U\})$

Example Recall $A_k = \bigcup_{\substack{0 \le i < 2^k \\ i \text{ even}}} \left(\frac{i}{2^k}, \frac{i+1}{2^k} \right]$, let $R_k = \mathbb{1}_{A_k}$, $R_k(x) = \begin{cases} 1 & \text{if } x \in A_k \\ 0 & \text{if } x \notin A_k \end{cases}$

indicator $f^{\underline{n}}$ of set A_k

Claim R_k is a real-valued random variable on $([0,1], \mathcal{B}([0,1]), \text{Leb}_{[0,1]})$

Proof Note $\{R_k = 1\} = A_k \in \mathcal{B}[0,1]$, $\{R_k = 0\} = A_k^c = [0,1] \setminus A_k \in \mathcal{B}[0,1]$

So for all $U \in \mathcal{B}(\mathbb{R})$, we have $\{R_k \in U\} = \begin{cases} [0,1] & \text{if } 0 \in U, 1 \in U \\ A_k & \text{if } 0 \in U, 1 \notin U \\ A_k^c & \text{if } 0 \in U, 1 \notin U \\ \phi & \text{if } 0 \notin U, 1 \notin U \end{cases}$

$[0,1] = \Omega \in \mathcal{B}([0,1])$
$A_k \in \mathcal{B}([0,1])$
$A_k^c \in \mathcal{B}[0,1]$
$\phi \in \mathcal{B}[0,1]$

$\square$

What about $R_1 + R_2$? Well,

$\{R_1 + R_2 = 2\} = A_1 \cap A_2 = (0, \tfrac{1}{4}]$
$\{R_1 + R_2 = 1\} = A_1 \triangle A_2 = (\tfrac{1}{4}, \tfrac{3}{4}]$
$\{R_1 + R_2 = 0\} = A_1^c \cap A_2^c = \{0\} \cup (3/4, 1]$

all in $\mathcal{B}[0,1]$

so $\{R_k \in U\} = \begin{cases} [0,1] & \text{if } 0 \in U, 1 \in U, 2 \in U \\ \vdots & \\ A_1 \triangle A_2 & \text{if } 0 \notin U, 1 \in U, 2 \notin U \\ \vdots & \\ \phi & \text{if } 0 \notin U, 1 \notin U, 2 \notin U \end{cases}$

So $R_1 + R_2$ is a r.v.

The next theorem helps us avoid doing this on a case-by-case basis.

Theorem Let $(R,\mathcal{R})$ and $(S,\mathcal{S})$ be measurable spaces and $\mathcal{A}\subset\mathcal{S}$ with $\sigma(\mathcal{A})=\mathcal{S}$.
If $X: R\to S$ satisfies that $X^{-1}(A)\in\mathcal{R}$ for all $A\in\mathcal{A}$ then X is $(\mathcal{R}/\mathcal{S})$-measurable.

Proof Let $\mathcal{B} := \{S\in\mathcal{S} : X^{-1}(S)\in\mathcal{R}\}$. Since $\mathcal{A}\subset\mathcal{B}$, if we prove $\mathcal{B}$ is a σ-field then $\mathcal{S}=\sigma(\mathcal{A})\subset\mathcal{B}$, so $\mathcal{B}=\mathcal{S}$ and thus X is $(\mathcal{R}/\mathcal{S})$-measurable.

Claim $\mathcal{B}$ is a σ-field.

$\boxed{\text{Proof}}$ ① $X^{-1}(\phi)=\phi\in\mathcal{R}$ so $\phi\in\mathcal{B}$.

② If $E\in\mathcal{B}$ then $X^{-1}(E^c)=\{X\notin E\}=\{r\in R: X(r)\notin E\}=X^{-1}(E)^c\in\mathcal{R}$ so $E^c\in\mathcal{B}$.

③ If $(E_n, n\geq 1)$ disjoint $\in\mathcal{B}$ then $X^{-1}(\bigcup_{n\geq1}E_n)=\bigcup_{n\geq1}X^{-1}(E_n)\in\mathcal{R}$ so $\bigcup_{n\geq1}E_n\in\mathcal{B}$. □ □

Examples

① $(\Omega,\mathcal{F},\mathbb{P})$ p.s., $X:\Omega\to\mathbb{R}$. If $\{X\leq r\}=X^{-1}((-\infty,r])\in\mathcal{F}$ for all $r\in\mathbb{R}$ then X is a real random variable.
(Because $\mathcal{B}(\mathbb{R})=\sigma(\{(-\infty,r], r\in\mathbb{R}\})$.)

② If R,S are topological spaces and $X:R\to S$ is continuous then X is $(\mathcal{B}(R)/\mathcal{B}(S))$-measurable ($\equiv$ Borel f^n)

$\boxed{\text{Proof}}$: Let $\mathcal{A}=\{U\subset S: X^{-1}(U)\in\mathcal{B}(\mathbb{R})\}$. Then $\mathcal{A}$ contains all open sets in S, so $\sigma(\mathcal{A})=\mathcal{B}(S)$. ■

③ If $X:\Omega\to\mathbb{R}$ r.v. and $f:\mathbb{R}\to\mathbb{R}$ is continuous then $f(X)$ is another r.v.
(Because for $U\in\mathcal{B}(\mathbb{R})$, $\{f(X)\in U\}=\{X\in f^{-1}(U)\}\in\mathcal{F}$ since $f^{-1}(U)\in\mathcal{B}(\mathbb{R})$)

Last class

For $(R, \mathcal{R}), (S, \mathcal{S})$ measurable spaces, an $(\mathcal{R}/\mathcal{S})$-measurable map is a function $f: R \to S$ s.t. $f^{-1}(U) \in \mathcal{R}$ for all $U \in \mathcal{S}$.

If R, S are topological spaces and $\mathcal{R} = \mathcal{B}(R)$, $\mathcal{S} = \mathcal{B}(S)$ then f is called Borel

If $\mathcal{A} \subset \mathcal{S}$ has $\sigma(\mathcal{A}) = \mathcal{S}$, and $X: R \to S$ has $X^{-1}(U) \in \mathcal{R}$ for all $U \in \mathcal{A}$, then X is $(\mathcal{R}/\mathcal{S})$-measurable.

④ Fix J finite, random variables $X_j : \Omega \to \mathbb{R}$ for $j \in J$. We can view $X_J = (X_j, j \in J)$ as a f^n $X_J : \Omega \to \mathbb{R}^J$.

Claim: X_J is $(\mathcal{F}/\mathcal{B}(\mathbb{R}^J))$-meas. (i.e. it is an $\mathbb{R}^J$-valued r.v.)

$\boxed{\text{Proof}}$ For any $(b_j, j \in J) \in \mathbb{R}^J$, have $\{X_J \in \prod_{j \in J} (-\infty, b_j]\} = \bigcap_{j \in J} \{X_j \le b_j\} \in \mathcal{F}$; and $\sigma\left(\left\{\prod_{j \in J}(-\infty, b_j] : (b_j, j \in J) \in \mathbb{R}^J\right\}\right) = \mathcal{B}(\mathbb{R}^J)$ ∎

Exercise: Basic operations on random variables yield random variables

If $(\Omega, \mathcal{F})$ is a measurable space and $X, Y, (X_n, n \geq 1)$ are $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable maps, then so are

- $\mathbb{1}_{X \geq 0}$
- $aX + bY$ (for $a, b \in \mathbb{R}$)
- $X \cdot Y$, $(X/Y) \cdot \mathbb{1}_{Y \neq 0}$

Moreover,

- $\frac{X}{Y}$*,
- $\sup_{n \geq 1} X_n$, $\inf_{n \geq 1} X_n$,
- $\limsup_{n \geq 1} X_n$, $\liminf_{n \geq 1} X_n$

are all extended real-valued random variables.

*Interpreting $\frac{x}{0} = \begin{cases} \infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$

Definition Let $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$, let $\mathcal{B}(\mathbb{R}^*) = \sigma(\{\text{open sets in } \mathbb{R}\} \cup \{(x, \infty], x \in \mathbb{R}\} \cup \{[-\infty, x), x \in \mathbb{R}\})$

An extended real-valued random variable is an $(\mathcal{F}/\mathcal{B}(\mathbb{R}^*))$-measurable map, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

If $X:\Omega\to\mathbb{R}^*$ is an extended random variable then $\{X=\infty\}=\bigcap_{n\in\mathbb{N}}\{X\geq n\}=\bigcap_{n\in\mathbb{N}}X^{-1}((n,\infty])$ 26

is an event; likewise, $\{X=-\infty\}$ is an event, so $\{X\in\mathbb{R}\}=(\{X=\infty\}\cup\{X=-\infty\})^c$ is an event,

so $X\,\mathbb{1}_{\{X\in\mathbb{R}\}}$ is a real random variable.

The exercise gives, for example, that with

$$R_k=\mathbb{1}_{A_k},\qquad S_n=\sum_{k=1}^n R_k\qquad S_n(x)=\{\#\text{ zeros in first } n \text{ bits of } x\}$$

We have $\liminf_{n\to\infty}\frac{S_n}{n}$ is a r.v., $\limsup_{n\to\infty}\frac{S_n}{n}$ is a r.v.

Proposition If $(X_n, n\geq 1)$ is a seq. of r.v.s on p.s. $(\Omega,\mathcal{F},\mathbb{P})$ then

$E:=\{\lim_{n\to\infty}X_n \text{ exists}\}=\{\omega\in\Omega:\lim_{n\to\infty}X_n(\omega)\text{ exists}\}$ is an event (is in $\mathcal{F}$).

<u>Proof</u> By homework ex. $\overline{X}:=\limsup_{n\to\infty}X_n$ is an x.r.v., $\underline{X}=\liminf_{n\to\infty}X_n$ is an x.r.v.

So
$$E_\infty:=\{\lim_{n\to\infty}X_n=\infty\}=\{\overline{X}=\infty\}\cap\{\underline{X}=\infty\}\text{ is an event}$$
$$E_{-\infty}=\{\lim_{n\to\infty}X_n=-\infty\}=\{\overline{X}=-\infty\}\cap\{\underline{X}=-\infty\}\text{ is an event}$$

Also, $E_{bd}=\{X_n \text{ is a bdd seq}\}=\{\overline{X}<\infty\}\cap\{\underline{X}>-\infty\}$ is an event

So $E_{fin}=\{\lim_{n\to\infty}X_n \text{ exists and is finite}\}=E_{bd}\cap\bigcap_{m\in\mathbb{N}}\{\overline{X}-\underline{X}<\frac{1}{m}\}$ is an event $\square$

Generated σ-fields

Given measurable space $(S, \mathcal{S})$ and a collection $(Y_i, i \in I)$ of f^ns $Y_i : R \to S$ of f^ns with common domain R,

define $\sigma(X_i, i \in I) := \sigma\left(\bigcup_{i \in I} \{X_i^{-1}(S), S \in \mathcal{S}\}\right) = \bigcap_{\substack{\mathcal{F} \text{ a } \sigma\text{-field over } R \\ \forall i \in I,\ X_i \text{ is } (\mathcal{F}/\mathcal{S})\text{-measurable}}} \mathcal{F}$.

This is called the σ-field generated by $(X_i, i \in I)$.

NB For a single f^n $X : R \to S$, $\sigma(X) = \{X^{-1}(S) ; S \in \mathcal{S}\}$ so $\sigma(X_i, i \in I) = \sigma\left(\bigcup_{i \in I} \sigma(X_i)\right)$.

Proposition If $\mathcal{A} \subseteq \mathcal{S}$ and $\sigma(\mathcal{A}) = \mathcal{S}$ then $\sigma(X_i, i \in I) = \sigma\left(\bigcup_{i \in I} \{X^{-1}(S), S \in \mathcal{A}\}\right)$

Proof $\supseteq$ Obvious $\subseteq$ For any fixed $i \in I$, RHS $\supset \sigma(X^{-1}(S), S \in \mathcal{A}) = \sigma(X_i)$ so

RHS $\supset \bigcup_{i \in I} \sigma(X_i)$ so RHS $\supset \sigma\left(\bigcup_{i \in I} \sigma(X_i)\right) = $ LHS. $\square$

Independence of random variables

Def Given random variables $(X_i, i\in I)$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ say $(X_i, i\in I)$ are (mutually) independent if for all $J\subset I$ finite, for any Borel sets $(B_j, j\in J)$ in $\mathbb{R}$,

$$\mathbb{P}(X_j\in B_j \text{ for all } j\in J) = \prod_{j\in J} \mathbb{P}(X_j\in B_j)$$

Equivalently, $(X_i, i\in I)$ are indep. iff the generated σ-algebras $(\sigma(X_i), i\in I)$ are independent.

Prop Given $(X_i, i\in I)$ real r.v.s on $(\Omega, \mathcal{F}, \mathbb{P})$, $(X_i, i\in I)$ are mutually indep. $\iff$ $\forall\, J\subset I$ finite, $\forall\, (b_j, j\in J)\in\mathbb{R}^J$, $\mathbb{P}(X_j\le b_j \text{ for all } j\in J) = \prod_{j\in J}\mathbb{P}(X_j\le b_j)$.

Proof: $(X_i, i\in I)$ are mutually indep. $\iff$ $(\sigma(X_i), i\in I)$ are indep $\iff$ $\forall\, J\subset I$ finite, $(\sigma(X_j), j\in J)$ indep.

And $\mathcal{P}_j := \{\{X_j\le r\}, r\in\mathbb{R}\}$ has $\sigma(\mathcal{P}_j) = \sigma(X_j)$, so

$(\sigma(X_j), j\in J)$ indep. $\iff$ $\mathbb{P}\left(\bigcap_{j\in J}\{X_j\le r_j\}\right) = \prod_{j\in J}\mathbb{P}(X_j\le r_j)$ for all $(r_j, j\in J)\in\mathbb{R}^J$. $\square$

Corollary Let $R_k = 1 - \mathbb{1}_{A_k} = \mathbb{1}_{A_k^c}$ from earlier. Then $(R_k, k\ge 1)$ are independent.

Proof For $n\in\mathbb{N}$, $b_1,\dots,b_n\in\mathbb{R}$, $\mathbb{P}(R_k\le b_k, 1\le k\le n) = 1^{\#\{k: b_k\ge 1\}}\cdot\left(\frac{1}{2}\right)^{\#\{k: b_k\in[0,1)\}}\cdot 0^{\#\{k: b_k<0\}} = \prod_{k=1}^{n}\mathbb{P}(R_k\le b_k)$ $\square$