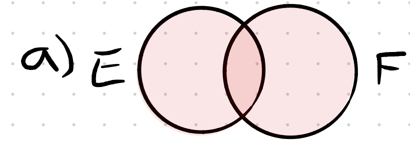
1. Measure Theory

Def: Fix a set Ω and a set $\mathcal{A}$ of subsets of Ω with $\emptyset \in \mathcal{A}$

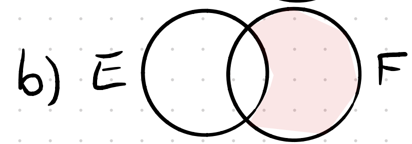
$\mathcal{A}$ is a **ring** if

a) If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$,



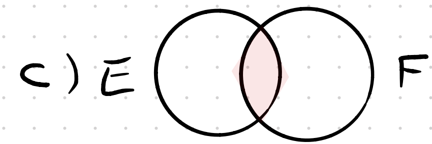
b) If $E, F \in \mathcal{A}$ then $F \setminus E \in \mathcal{A}$

$\mathcal{A}$ is a **π-system** if c) If $E, F \in \mathcal{A}$ then $E \cap F \in \mathcal{A}$



$\mathcal{A}$ is a **field** if it is a ring and also

b') If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$



Exercise Fields are π-systems.

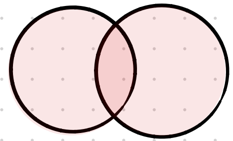
$\mathcal{A}$ is a **σ-field** if it is a field and also

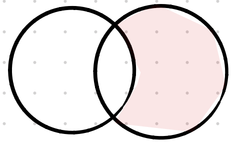
a') For any seq. $(A_n, n \geq 1)$ of elements of $\mathcal{A}$, $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.

Def: For any set $\mathcal{A}$ of subsets of Ω, the **σ-field generated by $\mathcal{A}$** is

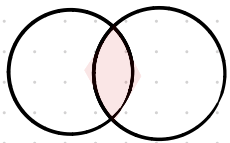
$$\sigma(\mathcal{A}) := \bigcap_{\{\mathcal{F} \supset \mathcal{A} \,:\, \mathcal{F} \text{ a } \sigma\text{-field}\}} \mathcal{F} .$$

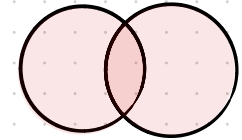
The definition in pictures

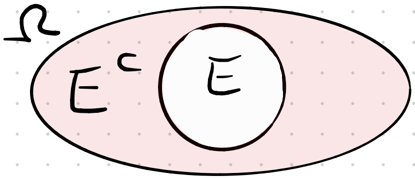
a) E  F

b) E  F

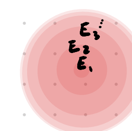
Ring

c) E  F π-system

$E \cup F$ E  F

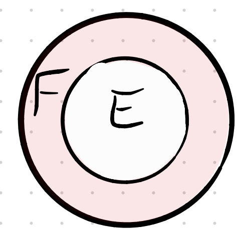
E^c Ω 

Field

E_∞ $E_\infty = \lim_{n \uparrow \infty} E_n$ 

σ-field

λ-system
(Ω must be an element)

$F \setminus E$ 

$$\mathcal{A} \to \sigma(\mathcal{A}) := \bigcap_{\{\mathcal{F} \supset \mathcal{A} :\, \mathcal{F}\ \sigma\text{-field}\}} \mathcal{F}$$

Pre-measure μ on ring $\mathcal{A}$: ① $\mu(\emptyset) = 0$;

② If $(A_n, n \geq 1)$ disj. elts of $\mathcal{A}$, $\bigcup_{n \geq 1} A_n \in \mathcal{A}$ then $\sum_{n \geq 1} \mu(A_n) = \mu\left(\bigcup_{n \geq 1} A_n\right)$

μ Additive:
$E \cap F = \emptyset \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F)$

μ countably add.
$(E_n, n \geq 1)$ disj $\Rightarrow \mu\left(\bigcup E_n\right) = \sum_n \mu(E_n)$

Building measures

Def Fix a ring $\mathcal{A}$ over Ω. A pre-measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0,\infty]$ with $\mu(\emptyset) = 0$ s.t.

for any seq. $(A_n, n \geq 1)$ of disjoint elements of $\mathcal{A}$,
if $\bigcup_{n\geq 1} A_n \in \mathcal{A}$ then $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$

We then say $(\Omega, \mathcal{A}, \mu)$ is a pre-measure space.

Carathéodory Extension Theorem Let $(\Omega, \mathcal{A}, \mu)$ be a pre-measure space. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{A}$ s.t. μ extends to a measure on $\mathcal{F}$.

Dynkin's Theorem: Let $(\Omega, \mathcal{F})$ be a set with a σ-field on it, and let $\mathcal{P} \subset \mathcal{F}$ be a π-system with $\sigma(\mathcal{P}) = \mathcal{F}$. If μ_1, μ_2 are measures on $\mathcal{F}$ and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{P}$ then $\mu_1 \equiv \mu_2$.

Aside

It seems to me that the following should be true.

Let $(\Omega, \mathcal{P})$ be a set with a π-system on it.

Let $\mu : \mathcal{P} \to [0, \infty)$ be s.t. $\mu(\emptyset) = 0$ and if

$(P_n, n \geq 1)$ are disjoint elements of $\mathcal{P}$ s.t. $\bigcup_{n \geq 1} P_n \overset{=}{\subseteq} Q \in \mathcal{P}$

then $\sum_{n \geq 1} \mu(P_n) \overset{=}{\leq} \mu(Q)$

[<u>Note</u>: If $Q \subseteq \bigcup_{n \geq 1} P_n$ then $Q = \bigcup_{n \geq 1} Q \cap P_n$]

Then there exists a measure on $\sigma(\mathcal{P})$ extending μ.

Examples

- $\Omega = \mathbb{R}$, $\mathcal{A}$ = Finite unions of intervals $(a,b] = \{(a_1,b_1] \cup \cdots \cup (a_k,b_k],\ a_1,\dots,a_k,b_1,\dots,b_k \in \mathbb{R}\}$

  (CDF: $F_X(b) - F_X(a) = \mathbb{P}(X \in (a,b]) =: \mu_X(a,b])$

  $\mathcal{A}$ is an algebra; want to know that F_X determines dist. of X.

- $\Omega = \{0,1\}^{\mathbb{N}} = \{(x_i, i \geq 1) : x_i \in \{0,1\}\ \forall i\}$.

  $\mathcal{A}$ = Cylinder sets. Cylinder set: for $S \subset \mathbb{N}$ finite and $\vec{y} = (y_i, i \in S)$,

  $$C_{\vec{y}} := \{\vec{x} \in \Omega : x_i = y_i\ \forall i \in S\}.$$

  For cylinder set $C_{\vec{y}}$ set $\mu(C_{\vec{y}}) = \left(\frac{1}{2}\right)^{|S|}$. ("IID Fair coins")

  Should be able to extend μ to a p.m. on $(\Omega, \sigma(\mathcal{A}))$; μ models "an ∞ sequence of fair coin tosses".

Carathéodory Proof

Idea: Approximate from above.

Let $(\Omega, \mathcal{A}, \mu)$ be a pre-measurable space

For $B \subset \Omega$ let

$$\mu^*(B) := \inf\Big(\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A},\ n\geq 1;\ \underbrace{\bigcup_{n\geq 1} A_n \supset B}_{\text{i.e. } (A_n,\, n\geq 1) \text{ covers } B.} \Big).$$

Prop: μ^* is an outer measure: $\mu^* : 2^{\Omega} \to [0,\infty]$ satisfies

i) $\mu^*(\emptyset) = 0$;

ii) $E \subset F \Rightarrow \mu(E) \leq \mu(F)$;

iii) if $(E_i, i\geq 1)$ are subsets of Ω then $\underbrace{\mu^*\Big(\bigcup_{i\geq 1} E_i\Big) \leq \sum_{i\geq 1} \mu^*(E_i)}_{\text{i.e. } \mu^* \text{ is subadditive.}}$.

Def: Given an outer measure μ^* on Ω, say $A \subset \Omega$ is μ^*-additive if for all $B \subset \Omega$, $\mu^*(B) = \mu^*(A\cap B) + \mu^*(A^c \cap B)$

Carathéodory Lemma Let $\mathcal{F} := \{A \subset \Omega : A \text{ is } \mu^*\text{-additive}\}$

Define $\mu : \mathcal{F} \to [0,\infty]$ by $\mu(B) = \mu^*(B)$. Then $(\Omega, \mathcal{F}, \mu)$ is a measure space

(i.e. $\mathcal{F}$ is a σ-algebra over Ω and μ is a measure on $\mathcal{F}$).

Proof of Prop

(i) If $B \in \mathcal{A}$ then $\mu^*(B) \leq \mu(B)$ since $(B, \phi, \phi, \ldots)$ covers B.

In particular $\mu^*(\phi) \leq \mu(\phi) = 0$ so $\mu^*(\phi) = 0$.

(ii) If $E \subset F$ then any cover of F is a cover of E so

$\mu^*(F)$ is an inf over a smaller set so $\mu^*(F) \geq \mu^*(E)$.

(iii) "Dyadic trick": Given $(E_i, i \geq 1)$ subsets of Ω. Write $E = \bigcup_{i \geq 1} E_i$

We prove: $\forall \varepsilon > 0$, $\mu^*(E) \leq \left(\sum_{i \geq 1} \mu^*(E_i)\right) + \varepsilon$.

Fix $\varepsilon > 0$, then for all $i \geq 1$ fix a cover $(A_n^i, n \geq 1)$ of E_i

s.t. $\sum_{n \geq 1} \mu(A_n^i) \leq \mu^*(E_i) + \frac{\varepsilon}{2^i}$.

Then $(A_n^i, n, i \geq 1)$ covers E so

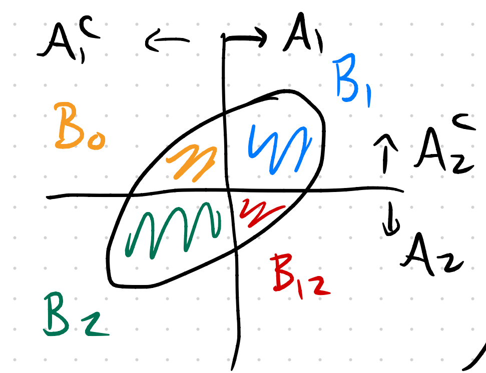
$$\mu^*(E) \leq \sum_{n, i \geq 1} \mu(A_n^i) \leq \sum_{i \geq 1} \left(\mu^*(E_i) + \frac{\varepsilon}{2^i}\right) = \left(\sum_{i \geq 1} \mu^*(E_i)\right) + \varepsilon \quad \square$$

Proof of Carathéodory Lemma

Step 1: prove $\mathcal{F}$ is a σ-field   Step 2: Prove μ is a measure on $\mathcal{F}$.

Step 1: ⋆ $\mathcal{F}$ obv. closed under complements (def. is invariant to $A \mapsto A^c$)

⋆ Closure under $\cap$ trickier. Fix $A_1, A_2 \in \mathcal{F}$ and any $B \subset \Omega$.



Write $B = B_0 \cup B_1 \cup B_2 \cup B_{12}$ according to $\cap$ with A_1, A_2.

Then

$$\mu^*(B) \overset{A_1 \in \mathcal{F}}{=} \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c)$$

$$\overset{A_2 \in \mathcal{F}}{=} \mu^*(B_1) + \mu^*(B_{12}) \quad \mu^*(B_0) + \mu^*(B_2)$$

Also

$$\mu^*(B \setminus B_{12}) = \mu^*(B_1) + \mu^*(B_0) + \mu^*(B_2)$$

So

$$\mu^*(B) = \mu^*(B \setminus B_{12}) + \mu^*(B_{12}) = \mu^*(B \cap (A_1 \cap A_2)^c) + \mu^*(B \cap A_1 \cap A_2)$$

so $A_1 \cap A_2 \in \mathcal{F}$.

* Countable $\cup$: Fix disjoint sets $(A_n, n \geq 1)$ in $\mathcal{F}$, let $A = \bigcup_{n \geq 1} A_n$ (8)

Fix $B \in \mathcal{F}$. Since μ^* is an outer measure, it is subadditive so

$$\mu(A) \leq \mu(A \cap B) + \mu(A^c \cap B); \text{ need to prove "}\geq\text{".}$$

"Cut A into pieces" with A_1, then A_2, etc. Disjointness plus the fact that all $A_i \in A$ gives

$$\mu^*(B) = \mu^*(A_1 \cap B) + \mu^*(A_2 \cap B) + \cdots + \mu^*(A_n \cap B) + \mu^*(B \cap \underbrace{\bigcap_{i=1}^{n} A_i^c}_{\cup\, A^c})$$

$B \cap \bigcap_{i=1}^{n} A_i^c \supseteq B \cap A^c$

$$\geq \mu^*(A_1 \cap B) + \cdots + \mu^*(A_n \cap B) + \mu^*(A^c \cap B)$$

Take a limit in n to get

$$\mu^*(B) \geq \sum_{i \geq 1} \mu^*(A_i \cap B) + \mu^*(A^c \cap B)$$

$$\geq \mu\Big(\big(\bigcup_{i \geq 1} A_i\big) \cap B\Big) + \mu^*(A^c \cap B) = \mu(A \cap B) + \mu(A^c \cap B), \text{ so } A \in \mathcal{F}.$$

We also just proved that if $(A_n, n\geq 1)$ disjoint sets in $\mathcal{F}$ then

$$\mu^*(A) \geq \sum_{i\geq 1} \mu^*(A_i \cap A) + \mu^*(A^c \cap A) \geq \sum_{i\geq 1} \mu^*(A_i)$$

so $\mu^*(A) = \sum_{i\geq 1} \mu^*(A_i)$, i.e. μ restricts to a measure on $\mathcal{F}$.

Proof of Carathéodory X Theorem Let μ^* be the outer measure as above.

Step 1: If $A \in \mathcal{A}$ then $\mu^*(A) = \mu(A)$ (so μ^* extends μ)

Step 2: $\mathcal{A} \subseteq \mathcal{F}$ (so $\mathcal{F}$ extends $\mathcal{A}$).

Step 1: We know $\mu^*(A) \leq \mu(A)$, want rev. ineq.

Let $(A_i, i\geq 1)$ be a cover of A. For $n \geq 1$ let $B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$

Then $(B_i, i\geq 1)$ is a disjoint cover of A $\left(\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n B_i \ \forall n \right)$,

$\bigcup_{i\geq 1} (B_i \cap A) = A$ so

$$\mu(A) = \sum_{i\geq 1} \mu(B_i \cap A) \leq \sum_{i\geq 1} \mu(B_i) \leq \sum_{i\geq 1} \mu(A_i).$$

Take inf over covers $(A_i, i\geq 1)$ of A to get

$\mu(A) \leq \mu^*(A)$ $\square$

Step 2: Need to show if $A \in \mathcal{A}$ then A is μ^*-additive: $\forall$
$B \subseteq \Omega$, $\quad \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B)$.

"$\leq$" easy (subadditivity)

"$\geq$" Fix $\varepsilon > 0$, fix a cover $(A_n, n \geq 1)$ of B with elements of $\mathcal{A}$

s.t. $\displaystyle\sum_{n \geq 1} \mu(A_n) \leq \mu^*(B) + \varepsilon$.

Then $(A \cap A_n, n \geq 1)$ covers $A \cap B$ and $(A^c \cap A_n, n \geq 1)$ covers $A^c \cap B$,

so

$$\mu^*(A \cap B) + \mu^*(A^c \cap B) \leq \sum_{n \geq 1} \mu^*(A \cap A_n) + \sum_{n \geq 1} \mu^*(A^c \cap A_n)$$

$$= \sum_{n \geq 1} \mu^*(A \cap A_n) + \mu^*(A^c \cap A_n)$$

$$= \sum_{n \geq 1} \mu^*(A_n) \leq \mu^*(B) + \varepsilon.$$

But $\varepsilon > 0$ arbitrary so

$$\mu^*(A \cap B) + \mu^*(A^c \cap B) \leq \mu^*(B) + \varepsilon. \quad \square$$

Last class:

Carathéodory Extension Theorem Let $(\Omega, \mathcal{A}, \mu)$ be a pre-measure space. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{A}$ s.t. μ extends to a measure on $\mathcal{F}$

The field $\mathcal{F}$ was the sets E s.t. $\mu(F) = \mu^*(E \cap F) + \mu^*(E^c \cap F)$ for all $F \subset \Omega$.

If $\mathcal{A} = \mathcal{A}(\mathbb{R}) := \{(a_1, b_1] \cup \cdots \cup (a_k, b_k],\ k \in \mathbb{N},\ -\infty < a_1 \leq b_1 \leq \cdots \leq a_k \leq b_k < \infty\}$

then $\mathcal{F}$ is called the Lebesgue measurable sets of $\mathbb{R}$; denote this by $\mathcal{L}(\mathbb{R})$.

NB: $\mathcal{L}(\mathbb{R})$ is <u>not</u> the smallest σ-field containing $\mathcal{A}$. The smallest is $\sigma(\mathcal{A}(\mathbb{R})) = \mathcal{B}(\mathbb{R})$, the Borel σ-field over $\mathbb{R}$. In fact, $\mathcal{L}(\mathbb{R})$ is the completion of $\mathcal{B}(\mathbb{R})$.

This class:

1. Dynkin's Theorem: Let $(\Omega, \mathcal{F})$ be a set with a σ-field on it, and let $\mathcal{P} \subset \mathcal{F}$ be a π-system with $\sigma(\mathcal{P}) = \mathcal{F}$. If μ_1, μ_2 are measures on $\mathcal{F}$ and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{P}$ then $\mu_1 \equiv \mu_2$.
2. Stieltjes Measures

One more definition:

A set $\mathcal{A} \subset 2^{\Omega}$ is a λ-system over $\mathcal{A}$ if $\Omega \in \mathcal{A}$ and

- $E, F \in \mathcal{A},\ E \subset F \Rightarrow F \setminus E \in \mathcal{A}$
- $E_n \in \mathcal{A},\ n \geq 1$ with $E_n \uparrow E_\infty \Rightarrow E_\infty \in \mathcal{A}$

Exercises

① $\mathcal{A}$ a σ-field over $\Omega \Rightarrow \mathcal{A}$ a λ-system over Ω

② If $\mathcal{A} \subset 2^{\Omega}$ is a π-system and a λ-system then $\mathcal{A}$ is a σ-field.

③ If $\{\Lambda_i, i \in I\}$ are λ-systems over Ω then $\bigcap_{i \in I} \Lambda_i$ is a λ-system over Ω.

Dynkin's π-system lemma Let $\mathcal{P}$ be a π-system over Ω. Then

$$\left[\sigma(\mathcal{P}) :=\right] \bigcap_{\substack{\{\mathcal{F} \supset \mathcal{P} :\, \mathcal{F} \text{ a } \sigma\text{-field} \\ \text{over } \Omega\}}} \mathcal{F} = \bigcap_{\substack{\{\mathcal{F} \supset \mathcal{P} :\, \mathcal{F} \text{ a } \lambda\text{-system} \\ \text{over } \Omega\}}} \mathcal{F} \left[=: \lambda(\mathcal{P})\right]$$

Proof: $\lambda(\mathcal{P}) \subseteq \sigma(\mathcal{P})$

By ③, σ-fields are λ-systems, so RHS is an $\bigcap$ of a larger collection of sets.

$\sigma(\mathcal{P}) \subseteq \lambda(\mathcal{P})$ We'll show $\lambda(\mathcal{P})$ is a π-system. Exercise ② then implies $\lambda(\mathcal{P})$ is a σ-field, so $\sigma(\mathcal{P}) = \bigcap_{\mathcal{F}\ \sigma\text{-field}} \mathcal{F} \subseteq \lambda(\mathcal{P})$

Remains to prove $\lambda(\mathcal{P})$ is a π-system

Proof Must prove: $E \cap F \in \lambda(\mathcal{P})$ for all $E, F \in \lambda(\mathcal{P})$

- Say $E \in \lambda(\mathcal{P})$ is cooperative if $E \cap P \in \lambda(\mathcal{P})$ for all $P \in \mathcal{P}$.
- Say $E \in \lambda(\mathcal{P})$ is helpful if $E \cap F \in \lambda(\mathcal{P})$ for all $F \in \lambda(\mathcal{P})$.

If we show all elements of $\lambda(\mathcal{P})$ are helpful then we are done.

Which sets are cooperative? • Ω is cooperative $\Omega \cap P = P \in \lambda(\mathcal{P})$ for all $P \in \mathcal{P}$.

- If $E \in \mathcal{P}$ then $E \cap P \in \mathcal{P}$ for all $P \in \mathcal{P}$ so E is cooperative

All elts of $\mathcal{P}$ cooperative

- If E, F coop. then $\forall P \in \mathcal{P}$, $E \cap P \in \lambda(\mathcal{P})$ and $F \cap P \in \lambda(\mathcal{P})$

If also $E \subset F$ then $E \cap P \subset F \cap P$, so $(F \setminus E) \cap P = (F \cap P) \setminus (E \cap P) \in \lambda(\mathcal{P})$ $\lambda(\mathcal{P})$ a λ-system

If E, F coop. and $E \subset F$ then $F \setminus E$ coop.

- If $(E_n, n\geq 1)$ increasing, $E_n \uparrow E_\infty$, all E_n cooperative, then for all $P \in \mathcal{P}$, $E_n \cap P \in \lambda(\mathcal{P})$, and $E_n \cap P \uparrow E_\infty \cap P$ so $E_\infty \in \mathcal{P}$.

If $E_n \uparrow E_\infty$, all E_n coop. then E_∞ coop

We have showed that {Cooperative sets} is a λ-system containing $\mathcal{P}$; so all sets in $\lambda(\mathcal{P})$ cooperative.

Which sets are helpful?

- Ω is helpful $\quad \Omega \cap F = F \in \lambda(\mathcal{P})$ for all $F \in \lambda(\mathcal{P})$.
- If $E \in \mathcal{P}$ then for all $E \in \lambda(\mathcal{P})$, E cooperative so $E \cap F \in \lambda(\mathcal{P})$; so E helpful.

All elts of $\mathcal{P}$ are helpful

- If E, F helpful then $\forall G \in \lambda(\mathcal{P})$, $E \cap G \in \lambda(\mathcal{P})$ and $F \cap G \in \lambda(\mathcal{P})$

If also $E \subset F$ then $E \cap G \subset F \cap G$, so $(F \setminus E) \cap G = (F \cap G) \setminus (E \cap G) \in \lambda(\mathcal{P})$ $\quad$ $\lambda(\mathcal{P})$ a λ-system

If E, F helpful and $E \subset F$ then $F \setminus E$ helpful

- Likewise, if $(E_n, n \geq 1)$ are helpful and $E_n \uparrow E_\infty$ then E_∞ helpful.

So {helpful sets} is a λ-system containing $\mathcal{P}$; so all sets in $\lambda(\mathcal{P})$ are helpful. So we are done. $\blacksquare$

Dynkin's Theorem: Let $(\Omega, \mathcal{F})$ be a set with a σ-field on it, and let $\mathcal{P} \subset \mathcal{F}$ be a π-system with $\sigma(\mathcal{P}) = \mathcal{F}$. If μ_1, μ_2 are measures on $\mathcal{F}$ and $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{P}$ then $\mu_1 \equiv \mu_2$.

Proof of Dynkin's Thm

Let $\Lambda = \{F \in \mathcal{F} : \mu_1(F) = \mu_2(F)\}$.

Then

- $\mathcal{P} \subset \Lambda$ by def.
- If $E, F \in \Lambda$, $E \subset F$ then $\mu_1(F \setminus E) = \mu_1(F) - \mu_1(E)$

$$= \mu_2(F) - \mu_2(E) = \mu_2(F \setminus E)$$

  so $\mu_1(F \setminus E) = \mu_2(F \setminus E)$. If $E, F \in \Lambda$, $E \subset F$ then $F \setminus E \in \Lambda$

- If $E_n \in \Lambda$, $n \geq 1$ and $E_n \uparrow E_\infty$ then $\mu_1(E_\infty) = \lim_{n \to \infty} \mu_1(E_n) = \lim_{n \to \infty} \mu_2(E_n) = \mu_2(E_\infty)$

  If $E_n \in \Lambda$, $n \geq 1$ $E_n \uparrow E_\infty$ then $E_\infty \in \Lambda$

Thus Λ is a λ-system containing $\mathcal{P}$, so $\Lambda \supseteq \sigma(\mathcal{P})$, so $\mu_1 \equiv \mu_2$ $\square$

Four notes about last class

(0) Examples of π-systems {Open sets}; {Intervals}; {Boxes}

(1) In hypothesis of Dynkin's Theorem we should assume that $\Omega \in \mathcal{P}$, or equivalently that $\mu_1(\Omega) = \mu_2(\Omega)$

(2) Def: Given a measurable space $(\Omega, \mathcal{F})$, a measure μ on $\mathcal{F}$ is σ-finite if there exist sets $(\Omega_n, n \geq 1)$ in $\mathcal{F}$ s.t. $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < \infty$ for all n.

σ-finiteness should also appear in the hyp. of Dynkin's Theorem

In that case, if $\mu_1(E) = \mu_2(E)$ and $\mu_1(F) = \mu_2(F)$, and $E \subset F$, then

$$\mu_1(F \setminus E) \overset{\text{additivity of } \mu_1}{=} \lim_{n\to\infty} \mu_1((F \setminus E) \cap \Omega_n) = \lim_{n\to\infty} \left(\mu_1(F \cap \Omega_n) - \mu_1(E \cap \Omega_n)\right)$$

$$\overset{\text{assumption}}{=} \lim_{n\to\infty} \left(\mu_2(F \cap \Omega_n) - \mu_2(E \cap \Omega_n)\right)$$

$$\overset{\text{additivity of } \mu_2}{=} \lim_{n\to\infty} \mu_2((F \setminus E) \cap \Omega_n) = \mu_2(F \setminus E)$$

(3) Recall: $\mathcal{B}(\mathbb{R}) := \sigma(\{U \subset \mathbb{R} : U \text{ open}\}) = \sigma(\mathcal{A}(\mathbb{R}))$ is called the Borel sets of $\mathbb{R}$.
Likewise $\mathcal{B}(\mathbb{R}^d) := \sigma(\{U \subset \mathbb{R}^d : U \text{ open}\})$.

Key example: Cumulative distribution f^ns / Stieltjes functions.

$F(x) = \lim_{y \downarrow x} F(y)$.

Def: A Stieltjes function is a f^n $F: \mathbb{R} \to \mathbb{R}$ which is non-decreasing and right-continuous.
It is a CDF if $F(-\infty) := \lim_{x \to -\infty} F(x) = 0$ and $F(\infty) := \lim_{x \to \infty} F(x) = 1$

Prop: Let F be a dist. function, and let $\mathcal{A} = \mathcal{A}(\mathbb{R}) := \{(a_1, b_1] \cup \cdots \cup (a_k, b_k],\ k \in \mathbb{N},\ -\infty < a_1 \leq b_1 \leq \cdots \leq a_k \leq b_k < \infty\}$.
Define $\mu = \mu_F$ by $\mu\left(\bigcup_{i=1}^{k} (a_i, b_i]\right) = \sum_{i=1}^{k} F(b_i) - F(a_i)$.
Then μ is a pre-measure on ring $\mathcal{A}$.

Proof: We prove this for the special case that $F(x) = x$ (i.e. for Lebesgue measure)
General case: same proof, more notation (in notes!).

Step 1: The definition makes sense.
Suppose $\bigcup_{i=1}^{n} (a_i, b_i] = \bigcup_{i=1}^{m} (c_i, d_i]$, where both sides are disjoint unions.
Then for $i \in [n]$, $j \in [m]$, let $S_{ij} = (a_i, b_i] \cap (c_j, d_j]$. If $S_{ij} \neq \emptyset$ write $S_{ij} = (\ell_{ij}, r_{ij}]$

Then $\sum_{i=1}^{n} (b_i - a_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} (r_{ij} - \ell_{ij}) = \sum_{j=1}^{m} (d_j - c_j)$

So def. makes sense

Step 2 μ is additive.

If $\left[\bigcup_{i=1}^{n}(a_i,b_i]\right] \cap \left[\bigcup_{i=1}^{m}(c_i,d_i]\right] = \phi$ then

$$\mu\left(\bigcup_{i=1}^{n}(a_i,b_i] \cup \bigcup_{i=1}^{m}(c_i,d_i]\right) = \sum_{i=1}^{n}(b_i-a_i) + \sum_{i=1}^{m}(d_i-c_i) = \mu\left(\bigcup_{i=1}^{n}(a_i,b_i]\right) + \mu\left(\bigcup_{i=1}^{m}(c_i,d_i]\right)$$

So μ is additive.

Step 3 : μ is a pre-measure.

We must show: if $L := \bigcup_{i=1}^{n}(a_i,b_i] = \bigcup_{i=1}^{\infty}(c_i,d_i]$, where both are disjoint unions,

then $\mu(L) = \sum_{i=1}^{n}(b_i-a_i) = \sum_{i=1}^{\infty}(d_i-c_i)$.

$\mu(L) \geq \sum_{i=1}^{\infty} d_i - c_i$. For all m, $L \supseteq \bigcup_{i=1}^{m}(c_i,d_i]$ so $\mu(L) \geq \mu\left(\bigcup_{i=1}^{m}(c_i,d_i]\right) = \sum_{i=1}^{m}(d_i-c_i)$

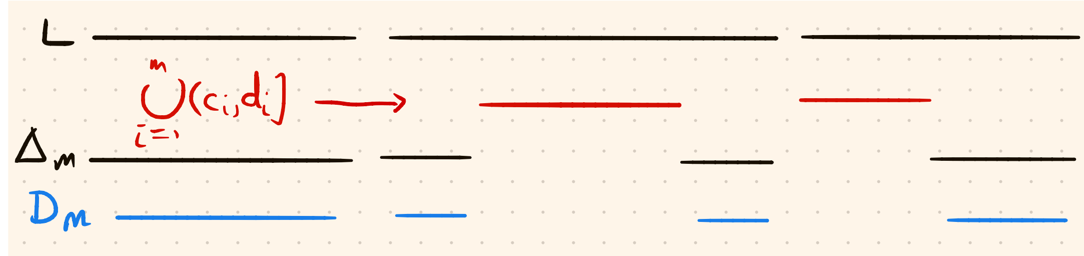
Thus $\mu(L) \geq \sum_{i=1}^{\infty}(d_i-c_i)$.

$\mu(L) \leq \sum_{i=1}^{\infty} d_i - c_i$. Suppose for a contradiction that $\mu(L) = \sum_{i=1}^{\infty}(d_i-c_i) + 2\varepsilon$, some $\varepsilon>0$.

We'll show the two sides weren't equal after all. For $m \geq 0$ write $\Delta_m = L \setminus \bigcup_{i=1}^{m}(c_i,d_i]$.

Then $\Delta_m = \bigcup_{i=1}^{n} (a_i,b_i] \setminus \bigcup_{i=1}^{m} (c_i,d_i] \in \mathcal{A}$ and $\Delta_m \supset \Delta_{m+1} \supset \cdots$

with $\Delta_m \downarrow 0$ as $m \to \infty$.



Also, $\mu(\Delta_m)$

$$= \mu\Big(L \setminus \bigcup_{i=1}^{m} (c_i,d_i]\Big)$$

$$= \mu(L) - \mu\Big(\bigcup_{i=1}^{m} (c_i,d_i]\Big) = \mu(L) - \sum_{i=1}^{m} (d_i - c_i) \geq 2\varepsilon.$$

Choose $D_m \in \mathcal{A}$ with $\overline{D}_m \subset \Delta_m$ s.t. $\mu(\Delta_m \setminus D_m) \leq \frac{\varepsilon}{2^m}$.

For $x \in \Delta_m$, if $\exists\, i$ s.t. $x \notin D_i$ then $x \in \Delta_i \setminus D_i$

Note that $\Delta_m = \bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_m \setminus D_i) \subseteq \bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_i \setminus D_i)$,

so

$$\mu(\Delta_m) \overset{\text{monotonicity}}{\leq} \mu\Big(\bigcap_{i=1}^{m} D_i \cup \bigcup_{i=1}^{m} (\Delta_i \setminus D_i)\Big)$$

$$\overset{\text{subadditivity}}{\leq} \mu\Big(\bigcap_{i=1}^{m} D_i\Big) + \underbrace{\sum_{i=1}^{m} \mu(\Delta_i - D_i)}_{\leq \sum_{i=1}^{m} \frac{\varepsilon}{2^i} < \varepsilon.}$$

$$< \mu\Big(\bigcap_{i=1}^{m} D_i\Big) + \varepsilon$$

Also, $\mu(\Delta_m) \geq 2\varepsilon$, so $\mu(\bigcap_{i=1}^{m} D_i) \geq \varepsilon$.

Thus $\bigcap_{i=1}^{m} \overline{D}_i \neq \phi$ for all m, so $\bigcap_{i=1}^{\infty} \overline{D}_i \neq \phi$. But $\bigcap_{i=1}^{\infty} \overline{D}_i \subseteq \bigcap_{i=1}^{\infty} \Delta_m = \phi$, a contradiction. $\square$

Theorem: Let F be a Stieltjes function. Then there exists a unique measure μ on $\sigma(\mathcal{A}(\mathbb{R}))$ s.t. $\mu(a,b] = F(b) - F(a)$ for all $-\infty < a \leq b < \infty$.

Proof: Existence

By the proposition, there exists a pre-measure μ on $\mathcal{A}(\mathbb{R})$ with this property.

By CXT, μ extends to a measure on $\mathcal{L}(\mathcal{A}(\mathbb{R})) \supseteq \sigma(\mathcal{A}(\mathbb{R}))$,

Uniqueness

Suppose μ_1, μ_2 are as in the Theorem statement.

Let $\mathcal{P} = \{(a,b] : -\infty < a \leq b < \infty\}$.

Then $\mathcal{P}$ is a π-system with $\sigma(\mathcal{P}) = \sigma(\mathcal{A}(\mathbb{R}))$, so by Dynkin's theorem $\mu_1 \equiv \mu_2$. $\square$

Exercise If μ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then there exists a Stieltjes f^n F s.t. $\mu_F = \mu$.