Measure and Integration.

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. (Throughout, unless otherwise specified.)

Our aim is to define $\int f \, d\mu = \int f(x) \, \mu(dx) = \mu(f)$ for as rich a class of functions $f : \Omega \to \mathbb{R}$ as possible. (Basically everything below also works for $f : \Omega \to \mathbb{R}^*$ unless "$\infty - \infty$" shows up.)

Definition. Say $f : \Omega \to \mathbb{R}$ is a simple function if for some $n \in \mathbb{N}$ there are sets $E_1, \dots, E_n \in \mathcal{F}$ and constants $c_1, \dots, c_n \in \mathbb{R}$ s.t. $f = c_1 \mathbb{1}_{E_1} + \cdots + c_n \mathbb{1}_{E_n}$.

$\mathbb{1}_E(\omega) = \begin{cases} 1, & \omega \in E \\ 0, & \text{otherwise} \end{cases}$

For such f, we define $\int f \, d\mu := \sum_{i=1}^{n} c_i \mu(E_i)$

Exercise: If f is simple then in fact there are disjoint sets $D_1, \dots, D_\ell \in \mathcal{F}$ and constants $b_1, \dots, b_\ell \in \mathbb{R}$ such that $f = \sum_{i=1}^{\ell} b_i \mathbb{1}_{D_i}$.

(Proof: For $S \subset [n]$ set $b_S = \sum_{i \in S} c_i$, $D_S = \bigcap_{i \in S} E_i \cap \bigcap_{i \in [n] \setminus S} E_i^c$)

Exercise: If $\sum_{i=1}^{n} c_i \cdot \mathbb{1}_{E_i} = \sum_{j=1}^{m} d_j \mathbb{1}_{F_j}$ for some $F_1, \dots, F_m \in \mathcal{F}$ and $d_1, \dots, d_m \in \mathbb{R}$ then

$$\sum_{i=1}^{n} c_i \mu(E_i) = \sum_{j=1}^{m} d_j \mu(F_j).$$

(So the definition makes sense.)

Definition For $E \in \mathcal{F}$, say E occurs μ-almost everywhere (or μ-a.e.) if $\mu(E^c) = 0$.

Proposition (Basic properties) Let $f,g:\Omega\to\mathbb{R}$ be simple.

① If $f\geq 0$ μ-a.e. then $\int f\,d\mu \geq 0$

② If $a\in\mathbb{R}$ then $\int af+g\,d\mu = a\int f\,d\mu + \int g\,d\mu$.

Proof ① Write $f=\sum_{i=1}^{n} c_i \mathbb{1}_{E_i}$ with $E_1,\dots,E_n$ disjoint. If some $c_i<0$ then since $f\geq 0$ μ-a.e. we must have $\mu(E_i)=0$. Thus $\int f\,d\mu = \sum_{i:c_i\geq 0} c_i\mu(E_i)\geq 0$.

② Write $g=\sum_{j=1}^{m} d_j \mathbb{1}_{F_j}$. Then $af+g=\sum_{i=1}^{n} c_i\mathbb{1}_{E_i}+\sum_{j=1}^{m} d_j\mathbb{1}_{F_j}$.

so $\int af+g\,d\mu = \sum_{i=1}^{n} c_i\mu(E_i)+\sum_{j=1}^{m} d_j\mu(F_j) = a\int f\,d\mu+\int g\,d\mu$. □

Corollary If f,g simple and $f\leq g$ a.e. then $\int f\,d\mu\leq\int g\,d\mu$.

Proof Here $g-f\geq 0$ a.e., so $0\leq\int g-f\,d\mu=\int g\,d\mu-\int f\,d\mu$. □

Definition Let $f:\Omega\to\mathbb{R}$ be non-negative and $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable. Then define

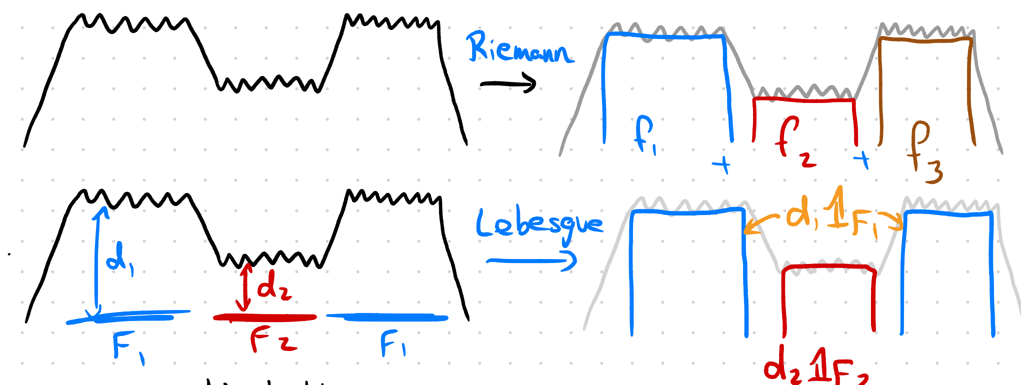
$$\int f\,d\mu = \sup_{\substack{g\leq f\\ g\text{ simple}}}\int g\,d\mu.$$

We call a simple function $g\leq f$ a *Lebesgue approximation* of f.

Remarks

① If f is simple then for $g \le f$ simple, $\int g \, d\mu \le \int f \, d\mu$ so $\sup_{\substack{g \le f \\ g \text{ simple}}} \int g \, d\mu = \int f \, d\mu$

② The definition of $\int f \, d\mu$ is a "horizontal" definition via lower approximation, whereas Riemann integration uses a "vertical" approximation.



Alternately, one may say that the Riemann approximation decomposes the domain; the Lebesgue approx. decomposes the range.

Finally, let $f : \Omega \to \mathbb{R}$ be $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable and write $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$. ← Exercise: these are measurable!

Definition If either $\int f^- d\mu < \infty$ or $\int f^+ d\mu < \infty$ then we define $\int f \, d\mu := \int f^+ d\mu - \int f^- d\mu$.

Exercise If $f \ge 0$ then $f^- = 0$ so $\int f^- = 0$ so this definition agrees with above one.

Notation If $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \text{Leb})$ we generally write $\int f(x)\,dx = \int f \,\text{Leb}(dx)$.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a p.s., $f: \Omega \to \mathbb{R}$ a r.v., we generally write $\mathbb{E}[f] = \int f \, d\mathbb{P}$.

Basic facts (All f's below $(\mathcal{F}/\mathcal{B}(\mathbb{R}))$-measurable.

Monotonicity If $f \leq g$ a.e. and $\int f \, d\mu$ and $\int g \, d\mu$ are defined then $\int f \, d\mu \leq \int g \, d\mu$

Monotone convergence theorem If $f_n \geq 0$ and $0 \leq f_n \uparrow f$ a.e. then $\int f_n \, d\mu \uparrow \int f \, d\mu$ [Midterm: what if $\int f_0^+ - \int f_0^- > -\infty$?]

(Beppo Levi's Thm? Billingsley 163?)

Linearity of Expectation For $f, g \geq 0$ and $a \in \mathbb{R}$, $\int af + g \, d\mu = a \int f \, d\mu + \int g \, d\mu$.

First do $f \geq 0$, and assume "everywhere" rather than "a.e."

Proofs **Monotonicity** Where $g^+ \geq f^+$ and $g^- \leq f^-$ so $\int g^+ d\mu - \int g^- d\mu \geq \int f^+ d\mu - \int f^- d\mu$. $\square$

Monotone convergence theorem For each n we have $f_n \leq f$ so $\int f_n d\mu \leq \int f \, d\mu$; thus $\lim_{n\to\infty} \int f_n d\mu \leq \int f d\mu$.

To prove the converse, fix any simple function $g = \sum_{i=1}^{m} c_i \mathbb{1}_{E_i}$ with $0 \leq g \leq f$ ($c_i > 0$, E_i disjoint)

Case 1: $\mu(E_i) = \infty$, some i

Then $\int f \, d\mu = \infty$. For $n \geq 1$ write $E_{i,n} = E_i \cap \{f_n > c_i/2\} = \{\omega \in E_i : f_n(\omega) \geq c_i/2\}$.

Then $E_{i,n} \uparrow E_i$ so $\mu(E_{i,n}) \uparrow \mu(E_i)$ so $\liminf_{n\to\infty} \int f_n \geq \liminf_{n\to\infty} \int \frac{c_i}{2} \mathbb{1}_{E_{i,n}} = \liminf_{n\to\infty} \frac{c_i}{2} \mu(E_{i,n}) = \frac{c_i}{2} \mu(E_i) = \infty$

Case 2: $\mu(E_i)<\infty$ for all i Fix $\varepsilon>0$, let $\delta<\varepsilon/\sum_{i=1}^m c_i\mu(E_i)$, and repeat the same idea.

For each $1\le i\le m$, each $n\ge 1$ let $E_{i,n}=E_i\cap\{f_n>c_i-\delta\}$. Then $f_n\ge \sum_{i=1}^m (c_i-\delta)\mathbb{1}_{E_{i,n}}$

So $$\int f_n \ge \sum_{i=1}^m (c_i-\delta)\mu(E_{i,n}) = \sum_{i=1}^m c_i\mu(E_{i,n}) - \delta\sum_{i=1}^m c_i\mu(E_{i,n}) \ge \sum_{i=1}^m c_i\mu(E_{i,n}) - \delta\sum_{i=1}^m c_i\mu(E_i)$$
$$> \sum_{i=1}^m c_i\mu(E_{i,n}) - \varepsilon$$

Moreover, $f_n\uparrow f$ so $E_{i,n}\uparrow E_i$ so

$$\sum_{i=1}^n c_i\mu(E_{i,n})\uparrow \sum_{i=1}^n c_i\mu(E_i).$$ Therefore $$\liminf_{n\to\infty}\int f_n\,d\mu > \sum_{i=1}^n c_i\mu(E_i)-\varepsilon = \int g\,d\mu-\varepsilon.$$

This holds for all $\varepsilon>0$ and all simple $0\le g\le f$, so $\liminf_{n\to\infty}\int f_n\,d\mu \ge \int f\,d\mu$ $\square$

↪ The "a.e." part is missing; we'll fill this in shortly.

Linearity of Integration We already proved this when f and g are simple. For general case, need a lemma.

Lemma Let $f\ge 0$ be measurable. Then there exist simple f_n's $(f_n, n\ge 1)$ s.t. $0\le f_n\uparrow f$.

Proof. For $0\le k<n\cdot 2^n$ let $B_{n,k}=\{\frac{k}{2^n}\le f<\frac{k+1}{2^n}\}$. Then set $f_n=\sum_{k=0}^{n\cdot 2^n}\frac{k}{2^n}\mathbb{1}_{B_{n,k}}$.

Then $f_n\le f$ and $f\mathbb{1}_{f\le n}\le f_n+\frac{1}{2^n}$. ← Picture

So $$\liminf_{n\to\infty} f_n \ge \liminf_{n\to\infty}\left(f\mathbb{1}_{f\le n}-\frac{1}{2^n}\right)=f.\ \square$$

Linearity of integration proof

Let $0 \le f_n \uparrow f$, $0 \le g_n \uparrow g$. Then $cf_n + g_n \uparrow cf + g$, so

$$\int cf+g \, d\mu = \lim_{n\to\infty} \int cf_n + g_n \, d\mu = \lim_{n\to\infty} \left(c\int f_n \, d\mu + \int g_n \, d\mu \right) = c\int f \, d\mu + \int g \, d\mu. \quad \square$$

Corollary $\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu$. $\square$

If $\int |f| d\mu < \infty$ we say f is μ-integrable, write $f \in L_1(\mu)$. If $\int f^+ < \infty$ or $\int f^- < \infty$, say $\int f d\mu$ is defined.

START HERE (Throught class, $(\Omega, \mathcal{F}, \mu)$ σ-finite and "measurable" means $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable unless otherwise specified.)

Exercises

- Show that if f, g are μ-integrable then $\int af + b = a\int f + \int b$.
- Let $f \ge 0$ measurable. Then $\int f d\mu = 0 \iff f = 0$ a.e.

Recall exercise.

NB
- Definition of integral for simple functions is "forced".
  - Any non-negative f^n is $\uparrow$ limit of simple f^ns so if we want $\int f = \lim_n \int f_n$ when $0 \le f_n \uparrow f$, then def. of integral for non-neg. f is forced.
  - And then general def. forced if we want linearity of $\int$.

Prop If $0 \leq f \overset{\mu\text{-a.e.}}{\leq} g$ then $\int f \, d\mu \leq \int g \, d\mu$.

Proof Write $\hat{g} = g + (f-g) \mathbb{1}_{f>g}$. Then $f \leq \hat{g}$ so $\int f \, d\mu \leq \int \hat{g} \, d\mu$. Also, g and $f - g \mathbb{1}_{f>g} \geq 0$

so $\int \hat{g} \, d\mu = \int g \, d\mu + \int (f-g) \mathbb{1}_{f>g} \, d\mu$.

But $(f-g) \mathbb{1}_{f>g} \overset{\text{a.e.}}{=} 0$ so RHS $= \int g \, d\mu$. $\square$

Corollary If $f \overset{\mu\text{-a.e.}}{=} g$ then $\int f \, d\mu = \int g \, d\mu$ (provided the integrals are defined).

Proof $f \overset{\text{a.e.}}{\leq} g$ and $g \overset{\text{a.e.}}{\leq} f$ $\square$

Corollary If $0 \leq f_n \uparrow f$ a.e. then $\int f_n \, d\mu \to \int f \, d\mu$.

Monotone convergence theorem

<u>Proof</u> Define the event $E = \{\omega \in \Omega : f_n(\omega) \uparrow f(\omega) \text{ as } n \to \infty\}$. By assumption $\mu(E^c) = 0$.

Now set $f_n' = f_n \cdot \mathbb{1}_E$ and $f' = f \mathbb{1}_E$.

Then $0 \leq f_n' \uparrow f'$ everywhere, so $\lim_{n\to\infty} \int f_n' \, d\mu = \int f' \, d\mu$.

But $f_n' \overset{\mu\text{-a.e.}}{=} f_n$ and $f' \overset{\mu\text{-a.e.}}{=} f$, so $\int f_n' \, d\mu = \int f_n \, d\mu$ and $\int f' \, d\mu = \int f \, d\mu$ $\square$

Exercise • If $(g_n, n \geq 1)$ non-negative, measurable, then $\sum_{n\geq 1} \int g_n \, d\mu = \int \sum_{n \geq 1} g_n \, d\mu$.

Fatou's Lemma If $(g_n, n \geq 1)$ non-negative, measurable then $\int \liminf_{n\to\infty} g_n \, d\mu \leq \liminf_{n\to\infty} \int g_n \, d\mu$

Proof By definition, $\liminf_{n\to\infty} g_n = \lim_{n\to\infty} (\inf_{m \geq n} g_m)$ and $\inf_{m\geq n} g_m \uparrow \liminf_{n\to\infty} g_n$, so

by M.C.T. $\int \liminf_{n\to\infty} g_n = \lim_{n\to\infty} \int \inf_{m \geq n} g_m = \text{RHS}$

By monotonicity, for all $p \geq n$, $\int \inf_{m\geq n} g_m \, d\mu \leq \int g_p \, d\mu$, so $\int \inf_{m \geq n} g_m \, d\mu \leq \inf_{p \geq n} \int g_p \, d\mu$,

so $\text{RHS} \leq \lim_{n\to\infty} \inf_{p\geq n} \int g_p \, d\mu = \liminf_{n\to\infty} \int g_n \, d\mu$ $\square$

$\underbrace{\int \liminf_{n\to\infty} X_n \, d\mathbb{P}}_{} \leq \underbrace{\liminf_{n\to\infty} \int X_n \, d\mathbb{P}}_{}$

In Probability: This says that for a sequence $(X_n, n \geq 1)$ of non-neg. r.v.s, $\mathbb{E} \liminf_{n\to\infty} X_n \leq \liminf_{n\to\infty} \mathbb{E} X_n$

Dominated Convergence Theorem

Suppose $f_n \xrightarrow{\mu\text{-a.e.}} f$. If there exists $g \in L_1(\mu)$ s.t. $|f_n| \leq g$ μ-a.e.

for all $n \geq 1$, then $\int f_n \, d\mu \longrightarrow \int f \, d\mu$.

Proof Can assume $f_n \to f$ everywhere and that $|f_n| \leq g$ everywhere (for all n).

Note $0 \leq g - f_n \leq 2g$ and $0 \leq g + f_n \leq 2g$ so $g - f_n, g + f_n \in L_1(\mu)$.

Now apply Fatou's lemma to $g + f_n$, $g - f_n$ to get

$$\int g + f \, d\mu = \int \liminf_{n\to\infty} (g + f_n) \, d\mu \leq \liminf_{n\to\infty} \int g + f_n \, d\mu \leq g + \liminf_{n\to\infty} \int f_n \, d\mu$$

$$\int g - f \, d\mu = \int \liminf_{n\to\infty} (g - f_n) \, d\mu \leq \liminf_{n\to\infty} \int g - f_n \, d\mu = g - \limsup_{n\to\infty} \int f_n \, d\mu$$

Subtract $\int g \, d\mu$ from both equations to get $\int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu \leq \limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$

So the limit must exist and equal $\int f \, d\mu$ $\square$

Corollary (Bounded convergence theorem). Suppose $\mu(\Omega) < \infty$. If $f_n \to f$ μ-a.e. and there is $M > 0$ s.t. $|f_n| \leq M$ for all $n \geq 1$ then $\int f_n \, d\mu \to \int f \, d\mu$.

$\boxed{\text{Proof}}$ Apply D.C.T. with $g \equiv M$. $\square$

Special Case: If $(X_n, n \geq 1)$ bounded r.v.s $X_n : \Omega \to \mathbb{R}$ and $X_n \xrightarrow{a.s.} X$ then $\mathbb{E}X_n \to \mathbb{E}X$.

Exercise If $(f_n, n \geq 1)$ measurable and $\sum_{n\geq 1} \int |f_n| \, d\mu < \infty$, then $\sum_{n\geq 1} f_n$ is μ-a.e. uniformly convergent and $\sum_{n\geq 1} \int f_n \, d\mu = \int \sum_{n\geq 1} f_n \, d\mu$.