Sep. 25 Lecture 9

Proposition If f, g are finite valued and measurable, then $f+g$, $f\cdot g$ are measurable

$f \vee g := \max\{f, g\}$. $f \wedge g := \min\{f, g\}$ are measurable

Proof. $\forall a \in \mathbb{R}$. $\{f+g<a\} = \{f<a-g\} = \bigsqcup_{q\in\mathbb{Q}} \{f<q<a-g\}$

$= \bigsqcup_{q\in\mathbb{Q}} \{f<q\} \cap \{g<a-q\} \in \mathcal{M}$

$\Rightarrow f+g$ is measurable $\Rightarrow f-g$ measurable $\Rightarrow$ $f\cdot g = \frac{1}{4}\left((f+g)^2-(f-g)^2\right)$ is measurable

$f \vee g = \frac{1}{2}\left(|f-g| + (f+g)\right)$ $\Rightarrow$ $f\vee g$ is measurable

$f \wedge g = -\max\{-f, -g\}$ $\Rightarrow$ $f \wedge g$ is measurable $\square$

Cor If f is measurable, then $f^+ := f \vee 0 = \max\{f, 0\}$ (positive part of f)

$f^- := -(f \wedge 0) = \max\{-f, 0\}$. (negative part of f, but taking nonnegative value!)

and $\forall K \in \mathbb{R}$ $f\wedge K := \min\{f, K\}$. $f \vee K := \max\{f, K\}$

are all measurable.

Remark: $|f| = f^+ + f^-$. $f = f^+ - f^-$ ("$\infty - \infty$" won't occur here. why?)

Proposition Let $\{f_n : n \geq 1\}$ be a sequence of measurable functions. Then,

$\sup_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ are all measurable functions

Recall that for $\{a_n : n\geq 1\} \subseteq \overline{\mathbb{R}}$. $\limsup_n a_n = \inf_{n\geq 1}(\sup_{m\geq n} a_m)$ and $\liminf_n a_n = \sup_{n\geq 1}(\inf_{m\geq n} a_m)$

So. $\forall x \in \mathbb{R}$. $\limsup_n f_n(x) = \limsup_n (f_n(x)) = \inf_{n\geq 1}(\sup_{m\geq n} f_m(x))$, the others are similar.

Proof: To show $\sup_n f_n$ is measurable. we verify that $\forall a \in \mathbb{R}$. $\{\sup_n f_n \leq a\} \in \mathcal{M}$

$x \in \{\sup_n f_n \leq a\} \iff \sup_n f_n(x) \leq a \iff f_n(x) \leq a \ \forall n \geq 1 \iff x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\}$

That is, $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \{f_n \leq a\} \in \mathcal{M}$. $\Rightarrow \sup_n f_n$ is measurable.

IF we had chosen to study $\{\sup_n f_n < a\}$:

NOTE that $\{\sup_n f_n < a\} \neq \bigcap_{n=1}^{\infty} \{f_n < a\}$ (e.g. $f_n = a - \frac{1}{n}$. $f_n < a$ $\forall n \geq 1$. but $\sup_n f_n = a$)

But $\{\sup_n f_n < a\} = \bigcup_{k=1}^{\infty} \{\sup_n f_n \leq a - \frac{1}{k}\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{f_n \leq a - \frac{1}{k}\}$

Exercise: Prove $\inf_n f_n$ is measurable

Given $a \in \mathbb{R}$, compare $\{\inf_n f_n < a\}$ and $\bigcup_{n=1}^{\infty} \{f_n < a\}$.

What about $\{\inf_n f_n \le a\}$ and $\bigcup_{n=1}^{\infty} \{f_n \le a\}$?

The measurability of $\limsup_n f_n$ and $\liminf_n f_n$ directly follows from the measurability of $\sup_n f_n$ and $\inf_n f_n$

For example. $\forall m \ge 1$. $g_m := \sup_{n \ge m} f_n$ is measurable $\Rightarrow \limsup_n f_n = \inf_{m \ge 1} g_m$ is measurable

Exercise. Express $\{\limsup_n f_n < a\}$ $\{\limsup_n f_n \le a\}$. $\{\liminf_n f_n < a\}$. $\{\liminf_n f_n \le a\}$

e.g. $\bigcup_{m \ge 1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \{f_n \le a - \frac{1}{k}\}$ $\qquad$ $\bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{f_n < a - \frac{1}{k}\}$

Proposition Let $\{f_n : n \ge 1\}$ be a sequence of measurable functions. Then, the following sets are in $\mathcal{U}$.

$\{\lim_{n\to\infty} f_n \text{ exists in } \mathbb{R}\}$ $\{\lim_{n\to\infty} f_n = \infty\}$. $\{\lim_{n\to\infty} f_n = -\infty\}$. $\{\lim_{n\to\infty} f_n = c\}$ $(\forall c \in \mathbb{R})$

Moreover, if $f := \lim_{n\to\infty} f_n$ exists (in $\mathbb{R}$ or as $\pm\infty$) a.e., then f is measurable. (Assignment 2)

§2.2 Approximation by Simple Functions

Let $f: \mathbb{R} \to \overline{\mathbb{R}}$ be a measurable function. We can "dissect" f in the following steps

1. $f = f^+ - f^-$. $f^+ := f \vee 0$. $f^- := -(f \wedge 0)$. both measurable non-negative.

2. $\forall n \geq 1$. set $f_n^+ := \underbrace{(f^+ \wedge n)}_{\text{truncation}} \underbrace{\mathbb{1}_{[-n,n]}}_{\text{cut off}}$ (i.e. $\forall x \in \mathbb{R}$ $f_n^+(x) = \mathbb{1}_{[-n,n]}(x)\,(f(x) \wedge n)$)

   $\forall n \geq 1$. f_n^+ is measurable. bounded and supported on a finite measure set.

   Observe that $f_n^+ \uparrow$ (i.e. $f_n^+ \leq f_{n+1}^+$) and $f^+ = \lim_{n\to\infty} f_n^+$

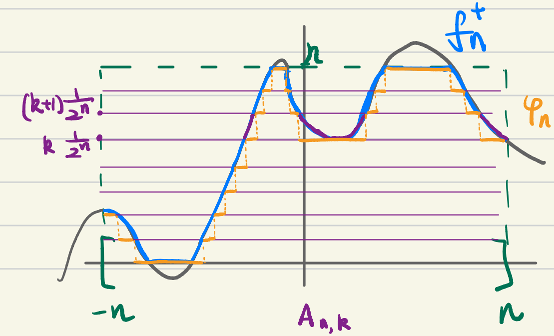
Similarly. $f_n^- = (f^- \wedge n)\mathbb{1}_{[-n,n]}$. same properties. and $f_n^- \uparrow$ s.t. $f^- = \lim_{n\to\infty} f_n^-$

3. $\forall n \geq 1$. Starting from f_n^+. define

for $k = 0, 1, 2, \cdots, 2^n \cdot n$. $A_{n,k} := \left\{ x \in [-n,n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n} \right\}$

Note: $k \neq l \Rightarrow A_{n,k} \cap A_{n,l} = \phi$. $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$

Set $\varphi_n := \sum_{k=0}^{n \cdot 2^n} \mathbb{1}_{A_{n,k}} \cdot \frac{k}{2^n}$. i.e. $\varphi_n(x) = \frac{k}{2^n}$ if $x \in A_{n,k}$



$\forall n \geq 1$, φ_n is measurable (because $A_{n,k} \in \mathcal{M}$ $\forall n$ $\forall k$), non-negative, bounded,

and $\varphi_n \uparrow$ (because $\forall x \in A_{n,k}$ $\varphi_n(x) = \frac{k}{2^n}$, but $\varphi_{n+1}(x) = \begin{cases} \frac{k}{2^n} & \text{if } x \in A_{n+1,2k} \\ \frac{2k+1}{2^{n+1}} & \text{if } x \in A_{n+1,2k+1} \end{cases}$)

(so $\lim_n \varphi_n$ exists)

$\varphi_{n+1}(x) \geq \varphi_n(x)$

and $\forall x \in \mathbb{R}$, $\varphi_n(x) \leq f_n^+(x)$ and $|f_n^+(x) - \varphi_n(x)| \leq \frac{1}{2^n}$

Moreover, φ_n is a simple function and $\lim_{n\to\infty} \varphi_n(x) = f^+(x)$ $\forall x \in \mathbb{R}$

Definition φ is called a simple function if φ takes the form of $\varphi = \sum_{k=1}^{L} a_k \mathbb{1}_{E_k}$

where $L \geq 1$, $\forall k = 1, 2, \cdots, L$ a_k is a constant. $E_k \in \mathcal{M}$ and $m(E_k) < \infty$.

Theorem If g is measurable and non-negative, then $\exists \{\varphi_n : n \geq 1\}$ a sequence of non-negative simple functions s.t. $\varphi_n \uparrow$ and $\lim_{n\to\infty} \varphi_n(x) = g(x)$ $\forall x \in \mathbb{R}$.

Next time: we will explain approximation by "step functions" in the "a.e. convergence" sense

We will also discuss "convergence a.e." v.s. "convergence in measure".