Sep. 20

Lecture 8

Chapter 2 Integration Theory

§2.1 Measurable Function

We consider function f defined on $\mathbb{R}$ (or $A \subseteq \mathbb{R}$). We will assume in general

f could take $\pm\infty$ value. i.e. $f: \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ (f extended-real-valued)

If $-\infty < f(x) < \infty$. $\forall x \in \mathbb{R}$. then we say f is finite valued.

$\forall a \in \mathbb{R}$. $f^{-1}([-\infty, a)) = \{x \in \mathbb{R} : f(x) < a\}$. (also written as $\{f<a\}$) is the inverse image of $[-\infty, a)$ under f

including "$f(x) = -\infty$"

Similarly. $\forall B \subseteq \overline{\mathbb{R}}$. $f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\}$

Definition: $f: \mathbb{R} \to \overline{\mathbb{R}}$ is a measurable function if $\forall a \in \mathbb{R}$. $\{f<a\}$ is measurable

Properties of measurable function

Proposition There are equivalent definitions of measurability

f is measurable $\iff \forall a \in \mathbb{R} \quad \{f \geq a\} \in \mathcal{M}$

$\iff \forall a \in \mathbb{R} \quad \{f > a\} \in \mathcal{M}$

$\overset{(\Delta)}{\iff} \forall a \in \mathbb{R} \quad \{f \leq a\} \in \mathcal{M}$

(To see (Δ), note that $\forall a \in \mathbb{R}$, $\quad \{f<a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\}$ and $\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\}$)

If f is finite valued, then f is measurable $\iff \forall a, b \in \mathbb{R} \quad f^{-1}([a,b)) \in \mathcal{M}$

$\iff \forall a, b \in \mathbb{R} \quad f^{-1}([a,b]) \in \mathcal{M}$

$\iff \forall a, b \in \mathbb{R} \quad f^{-1}((a,b)) \in \mathcal{M}$

$\iff \forall a, b \in \mathbb{R} \quad f^{-1}((a,b]) \in \mathcal{M}$

Consider the Borel σ-algebra of subsets of $\overline{\mathbb{R}}$: $\quad \mathcal{B}_{\overline{\mathbb{R}}} := \sigma\left(\mathcal{B}_{\mathbb{R}} \cup \{\{\infty\}, \{-\infty\}\}\right)$

Verify that $\quad \mathcal{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ (note that the complement "A^c" is $\overline{\mathbb{R}} \setminus A$)

Clearly, $\forall a \in \mathbb{R} \quad [-\infty, a) = \{-\infty\} \cup (-\infty, a) \in \sigma\left(\mathcal{B}_{\mathbb{R}} \cup \{\{\infty\}, \{-\infty\}\}\right) \Rightarrow \sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$

Conversely, $\quad \{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n)$, $\{\infty\} = \overline{\mathbb{R}} \setminus \left(\bigcup_{n=1}^{\infty} [-\infty, n)\right) \quad \Rightarrow \{\infty\}, \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$

$\forall a \in \mathbb{R}$, $(-\infty, a) = [-\infty, a) \setminus \{-\infty\}$. $\Rightarrow \mathcal{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$

$\Rightarrow \mathcal{B}_{\mathbb{R}} \cup \{\{\infty\}, \{-\infty\}\} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \quad \Rightarrow \quad \mathcal{B}_{\overline{\mathbb{R}}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$

Proposition $f: \mathbb{R} \to \overline{\mathbb{R}}$ is measurable $\iff \forall B \in \mathcal{B}_{\overline{\mathbb{R}}},\ f^{-1}(B) \in \mathcal{M}$

Proof: "$\Leftarrow$" is obvious. To see "$\Rightarrow$", use the result from a problem in Assignment 2.

Given $\mathcal{E}$ a collection of subsets of $\overline{\mathbb{R}}$. $f^{-1}(\mathcal{E}) := \{A \subseteq \mathbb{R} : A = f^{-1}(B) \text{ for some } B \in \mathcal{E}\}$

Then $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra (of subsets of $\mathbb{R}$) and $f^{-1}(\sigma(\mathcal{E})) = \sigma(f^{-1}(\mathcal{E}))$

Take $\mathcal{E} := \{[-\infty, a) : a \in \mathbb{R}\}$. Then $f^{-1}(\mathcal{B}_{\overline{\mathbb{R}}}) = \sigma(\{f^{-1}([-\infty, a)) : a \in \mathbb{R}\}) \subseteq \mathcal{M}$ □

Similarly, we have **Proposition**: $f: \mathbb{R} \to \mathbb{R}$ (i.e. f is finite valued!) is measurable $\iff \forall B \in \mathcal{B}_{\mathbb{R}},\ f^{-1}(B) \in \mathcal{M}$.

Proposition: Given $f: \mathbb{R} \to \overline{\mathbb{R}}$, define $f_{\mathbb{R}}(x) = \begin{cases} f(x) & \text{if } -\infty < f(x) < \infty \\ 0 & \text{otherwise.} \end{cases}$

f is measurable $\iff \forall B \in \mathcal{B}_{\mathbb{R}}\ \ f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$ AND $\{f = \infty\} \in \mathcal{M}$ $\{f = -\infty\} \in \mathcal{M}$

Proof: "$\Rightarrow$" follows immediately from the previous proposition.

"$\Leftarrow$" Assume the RHS. $\forall a \in \mathbb{R}$ $f^{-1}([-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)) \in \mathcal{M}$. □

Definition. If a statement is true for every $x \in A$ where $A \in \mathcal{M}$ and $m(A^c) = 0$,

then we say the statement is true a.e. (or "true for a.e. x") (a.e. = "almost every" or "almost everywhere")

(Assignment)

Proposition If f is measurable and $g = f$ a.e. ($g(x) = f(x)$ for a.e. $x \in \mathbb{R}$), then g is measurable

Cor If f is finite valued a.e., then f is measurable $\iff \forall a, b \in \mathbb{R}.\ f^{-1}((a,b)) \in \mathcal{M}$,

Proposition If $f = c$ (i.e. f is constant), then f is measurable. If $f = \mathbb{1}_A$ for some $A \subseteq \mathbb{R}$. (i.e., f is the characteristic function of A), then f is measurable $\iff A \in \mathcal{M}$

Proof: If $f \equiv c$, then $\forall a \in \mathbb{R}$ $\quad f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a > c \\ \phi & \text{if } a \leq c \end{cases} \in \mathcal{M}$

If $f = \mathbb{1}_A$, then $\forall a \in \mathbb{R}$. $\quad f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a > 1 \\ A^c & \text{if } 0 < a \leq 1 \\ \phi & \text{if } a \leq 0 \end{cases} \in \mathcal{M} \iff A \in \mathcal{M}$. $\square$

Proposition If f is a finite valued and continuous function on $\mathbb{R}$, then f is measurable

Proof: $f: \mathbb{R} \to \mathbb{R}$ is continuous $\iff \forall G \subseteq \mathbb{R}$ open. $f^{-1}(G)$ is open

$\Rightarrow \forall a, b \in \mathbb{R}$ $\quad f^{-1}((a,b))$ is open and hence in $\mathcal{M}$ $\square$

In fact, if $f: \mathbb{R} \to \mathbb{R}$ is continuous, then $\forall B \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ (i.e., $f^{-1}(\mathcal{B}_{\mathbb{R}}) \subseteq \mathcal{B}_{\mathbb{R}}$)
and if $f^{-1}: \mathbb{R} \to \mathbb{R}$ exists and is continuous, then $\forall B \in \mathcal{B}_{\mathbb{R}}$, $f(B) \in \mathcal{B}_{\mathbb{R}}$

Proposition If f is measurable, then $\forall c \in \mathbb{R}$, cf is measurable, (in particular, $-f$ is measurable). $|f|$ is measurable, and $\forall k \in \mathbb{N}$, f^k is measurable

Proof: Assume $c \neq 0$. Otherwise, $cf \equiv 0$ constant is measurable

$$\forall a \in \mathbb{R},\ (cf)^{-1}([-\infty, a)) = \begin{cases} f^{-1}([-\infty, a/c)) & \text{if } c>0 \\ f^{-1}((a/c, +\infty]) & \text{if } c<0 \end{cases} \in \mathcal{M}$$

$$\forall a \in \mathbb{R},\ |f|^{-1}([-\infty, a)) = \begin{cases} (-a, a) & \text{if } a>0 \\ \phi & \text{if } a \leq 0 \end{cases} \in \mathcal{M}$$

$$\forall a \in \mathbb{R}\quad (f^k)^{-1}([-\infty, a)) = \begin{cases} f^{-1}([-\infty, a^{1/k})) & \text{if } k \text{ is odd} \\ \phi & \text{if } k \text{ is even and } a \leq 0 \\ f^{-1}((-a^{\frac{1}{k}}, a^{\frac{1}{k}})) & \text{if } k \text{ is even and } a>0 \end{cases} \in \mathcal{M}.$$

$\square$

Proposition If f is finite valued and measurable and $g: \mathbb{R} \to \mathbb{R}$ is continuous

then $g \circ f$ is measurable.

Proof: $\forall a \in \mathbb{R}$. $(g \circ f)^{-1}((-\infty, a)) \overset{(*)}{=} f^{-1}(\underbrace{g^{-1}((-\infty, a))}_{\text{open}}) \in \mathcal{M}$.

To see $(*)$: $\forall B \subseteq \mathbb{R}$. $x \in (g \circ f)^{-1}(B) \iff g(f(x)) \in B$

$\iff f(x) \in g^{-1}(B)$

$\iff x \in f^{-1}(g^{-1}(B))$

Next time: We will continue with properties of measurable functions

(in particular, sequence / sum / product ... of measurable functions)

and discuss approximation by simple functions