Sep 18

Lecture 7

The Cantor set C: $\quad C \ni x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n} \xrightarrow{f} f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2} \frac{1}{2^n} \in [0,1]$ <u>surjective</u>

$a_n \in \{0,2\}$ base-3 $\qquad \frac{a_n}{2} \in \{0,1\}$ binary

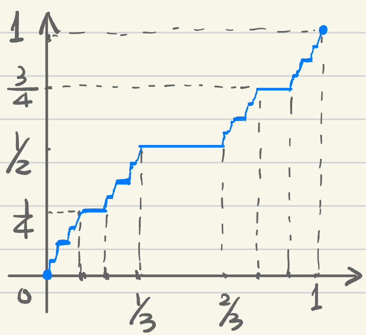
We will take a step further to extend the function f above to $f: [0,1] \to [0,1]$

<u>Definition</u> The Cantor-Lebesgue function is defined as

$$\forall x \in [0,1], \quad f(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \frac{1}{2^n} & \text{if } x \in C \text{ and } x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n} \text{ for } a_n \in \{0,2\}\ \forall n \geq 1 \\ \sup\{f(y) : y \in C,\ y < x\} & \text{if } x \notin C \end{cases}$$

(if $x \in (a,b)$ where (a,b) is removed in the construction of C, then $f(x) = f(a)$)

(Devil's staircase)



- $f(0) = 0$, $f(1) = 1$, $f \equiv \frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, $f \equiv \frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, $f \equiv \frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$ ...
- $f: [0,1] \to [0,1]$ is surjection
- f is non-decreasing (follows directly from definition)
- f is continuous.

Proof of continuity: $[0,1] \setminus C$ is a countable union of disjoint open intervals, and f is constant on each.

$\Rightarrow f$ is continuous on $[0,1] \setminus C$ and one-sided continuous at end points of each interval

Next, given $x \in C$, $\forall n \geq 1$, $\exists\ x_n, x_n' \in [0,1]$ s.t. $x_n < x < x_n'$ (unless x is 0 or 1 in which case only one-sided continuity is concerned) s.t. $f(x_n') - f(x_n) \leq \frac{1}{2^n}$. $\Rightarrow$ Continuity at x follows from monotonicity.

Q2: Is there $A \subseteq \mathbb{R}$. s.t. $A \notin \mathcal{U}$?

Yes. We will construct a non (Lebesgue) measurable set under the following axiom:

Axiom of Choice: If Σ is a collection of nonempty sets, then there exists a function $S: \Sigma \to \bigsqcup_{A \in \Sigma} A$ (called a selection function) s.t. $\forall A \in \Sigma$. $S(A) \in A$. (i.e., S maps every set in Σ to an element of that set)

Construction of the non measurable set:

Consider an equivalence relation on $[0,1]$: $a, b \in [0,1]$, "$a \sim b$" if and only if $a - b \in \mathbb{Q}$

(this is an equivalence since $a \sim a$, $a \sim b \Leftrightarrow b \sim a$, $a \sim b$ and $b \sim c \Rightarrow a \sim c$)

$E_a :=$ equivalence class of a

$\Sigma :=$ collection of all the equivalence classes. By the axiom of choice, we can select exactly one element s_a from E_a for each $E_a \in \Sigma$. We call s_a the representative of E_a

Proposition Set $N := \{s_a : s_a \text{ is the representative of } E_a,\ E_a \in \Sigma\}$. Then, N is non measurable.

Proof: We prove it by contradiction. Assume N is measurable. Let $\{q_k : k \geq 1\}$ be an enumeration of all the elements in $[-1, 1] \cap \mathbb{Q}$, and set $N_k := N + q_k$ (translation of N by q_k)

Observe that if $k \neq l$, then $N_k \cap N_l = \phi$. Assume otherwise: $\exists\ q_k \neq q_l$ $s_a, s_b \in N$ s.t. $q_k + s_a = q_l + s_b \Rightarrow s_a - s_b \in \mathbb{Q} \Rightarrow s_a$ and s_b are in the same equivalence class

$\Rightarrow s_a = s_b \Rightarrow q_k = q_l$. Contradiction.

Furthermore, if $x \in [0,1]$, then $x \sim s_a$ for some $s_a \in N$, and hence $x - s_a \in \mathbb{Q}$

Since $|x - s_a| \in [-1, 1]$, $x = s_a + q_k$ for some $k \geq 1 \Rightarrow x \in N_k$ for some $k \geq 1$

We conclude that $[0,1] \subseteq \bigcup_{k=1}^{\infty} N_k$ Meanwhile, by default, $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$

Finally, if N is measurable, then N_k is measurable and $m(N_k) = m(N)$ $\forall k \geq 1$

Since N_k's are disjoint. $\sum_{k=1}^{\infty} m(N_k) = m(\bigcup_{k=1}^{\infty} N_k) \in [1, 3]$ $\left(l([0,1]) \leq m(\bigcup_{k=1}^{\infty} N_k) \leq l([-1,2])\right)$

But this is not possible (neither $m(N) = 0$ or $m(N) > 0$ would be possible). Contradiction.

The non measurable set N constructed above is known as a Vitali set

The following theorem says that such non measurable sets are ubiquitous.

Theorem: For every $A \in \mathcal{M}$ with $m(A) > 0$, $\exists\, B \subseteq A$ s.t. B is non measurable.

Proof: Assume otherwise: $\exists A \in \mathcal{M}$ with $m(A) > 0$ and $\forall B \subseteq A$, $B \in \mathcal{M}$

Since $A = \bigsqcup_{n\in\mathbb{Z}} (A \cap [n, n+1])$, $m(A \cap [n, n+1]) > 0$ for some $n \in \mathbb{Z}$. $\Rightarrow m(A \cap [n, n+1] - n) > 0$.

Set $A' := A \cap [n, n+1] - n$. $A' \subseteq [0,1]$. $m(A') > 0$. $\forall B' \subseteq A'$, $B' + n \subseteq A$. So $B' + n \in \mathcal{M}$ by hypothesis.

So, W.L.O.G., we can assume $A \subseteq [0,1]$. $m(A) > 0$ and $\forall B \subseteq A$. $B \in \mathcal{M}$. N_k's are disjoint $\quad [0,1] \subseteq \bigsqcup_{k=1}^{\infty} N_k$

Let N, $\{q_k : k \geq 1\}$, $\{N_k : k \geq 1\}$ be the same as above. Set $A_k = A \cap N_k$ $\forall k \geq 1$. Then, A_k's are disjoint and $A = \bigsqcup_{k=1}^{\infty} A_k$

By the assumption, $A_k \in \mathcal{M}$ $\forall k \geq 1 \Rightarrow m(A) = \sum_{k=1}^{\infty} m(A_k) \Rightarrow m(A_k) > 0$ for some $k \geq 1$. Fix this k.

Set $L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}$. L is countably infinite. and $\{q_\ell + A_k : \ell \in L\}$ is disjoint ($q_\ell + q_k = q_{\ell'}$ for a unique ℓ')

Since $\bigsqcup_{\ell\in L} (q_\ell + A_k) \subseteq [-1, 2]$, $\sum_{\ell\in L} m(q_\ell + A_k) = m\left(\bigsqcup_{\ell\in L} (q_\ell + A_k)\right) \leq 3 \Rightarrow m(A_k) = 0$. Contradiction.

Q3: There exists $A \in \mathcal{M}$ but $A \notin \mathcal{B}_{\mathbb{R}}$. We will find such a set among the subsets of the Cantor set.

Let $f: [0,1] \to [0,1]$ be the Cantor-Lebesgue function defined as above.

Set $g(x) = f(x) + x$ for $x \in [0,1]$ $\Rightarrow$ g is continuous and strictly increasing

$\Rightarrow$ $g: [0,1] \to [0,2]$ is a bijection (strict increasing $\Rightarrow$ injective; $g(0)=0$, $g(1)=2$, continuity $\Rightarrow$ surjective)

$\Rightarrow$ $g^{-1}: [0,2] \to [0,1]$ exists and is continuous

In other words, $g: [0,1] \to [1,2]$ is a homeomorphism, and hence g maps open sets to open sets and closed sets to closed sets (recall that a function is continuous $\Leftrightarrow$ inverse image of an open set is open)

Furthermore, if $A \subseteq [0,1]$ and $A \in \mathcal{B}_{\mathbb{R}}$, then $g(A) \in \mathcal{B}_{\mathbb{R}}$ (direct consequence of an assignment problem
$g(\mathcal{B}_{\mathbb{R}}) = g(\sigma(\{\text{open sets}\})) = \sigma(g(\{\text{open sets}\}))$
$= \sigma(\{\text{open sets}\}) = \mathcal{B}_{\mathbb{R}}$)

Theorem There exists $A \subseteq C$ s.t. $A \in \mathcal{M}$ but $A \notin \mathcal{B}_{\mathbb{R}}$

Proof: Consider $g(C)$: the image of the Cantor set under g. Since C is closed, $g(C)$ is closed $\Rightarrow g(C) \in \mathcal{M}$.

Observe that if (a,b) is an open interval removed during the construction of C, i.e. if

$(a,b) \subseteq [0,1] \setminus C$ ($[0,1]\setminus C$ is a countable disjoint union of such intervals)

then c is constant on (a,b) and hence g maps (a,b) to an interval of the length $b-a$ (same length as (a,b)) $\Rightarrow m(g([0,1]\setminus C)) = m([0,1]\setminus C) = 1$.

Therefore, $m(g(C)) = m([0,2]\setminus g([0,1]\setminus C)) = 2-1 = 1$

Since $m(g(C)) > 0$, by the theorem related to Q2, $\exists B \subseteq g(C)$ s.t. $B \notin \mathcal{M}$

Set $A := g^{-1}(B)$. Then, $A \subseteq C$ and $m(C) = 0$, so $A \in \mathcal{M}$ and $m(A) = 0$.

We claim that $A \notin \mathcal{B}_{\mathbb{R}}$. Assume otherwise. $A \in \mathcal{B}_{\mathbb{R}} \Rightarrow g(A) \in \mathcal{B}_{\mathbb{R}}$.

However, $g(A) = g(g^{-1}(B)) = B \notin \mathcal{M}$. Contradiction.

We have finished "Chapter 1: Measure Theory". Will start "Integration Theory"

Next lecture: Definition of measurable function. Equivalent definitions of measurability.
Properties of measurable functions (in particular, properties of sequence of measurable functions)