Sep 13. Lecture 6

of $(*)$ Proof: Let $\mu$ be a measure that is translation invariant and finite on compact sets.

Assume $\mu((0,1]) = c \in [0,\infty)$. By finite additivity and translation invariance.

$\forall n \geq 1$. $\forall m = 1, 2, \cdots, n$, $\mu((\frac{m-1}{n}, \frac{m}{n}]) = \frac{1}{n}c$ and $\mu((0, \frac{m}{n}]) = \frac{m}{n}c$

$\Rightarrow \forall p, q \in \mathbb{Q}$, $p<q$. $\mu((p,q]) = (q-p)c$. By continuity of $\mu$, we obtain that

$\forall$ interval $I \subseteq \mathbb{R}$, $\mu(I) = \ell(I) \cdot c$, and $\forall x \in \mathbb{R}$. $\mu(\{x\}) = 0$.

In particular, for each $n \geq 1$. $\forall a, b \in \mathbb{R}$, $a<b$. $\mu((a,b) \cap [-n,n]) = c \cdot m((a,b) \cap [-n,n])$

where $c \cdot m$ is the measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ given by $c \cdot m(A) = c \cdot (m(A))$.

By the theorem above, $\mu = c \cdot m$ on $\mathcal{B}_{[-n,n]}$. Proceed similarly as in the previous proof,

we conclude that $\mu = c \cdot m$ on $\mathcal{B}_{\mathbb{R}}$.

**Proposition**: $m$ has the scaling property that $\forall A \in \mathcal{M}$. $\forall c \in \mathbb{R}$. $\underset{\{cx : x \in A\}}{c \cdot A} \in \mathcal{M}$ and $m(cA) = |c| \, m(A)$

In particular, $m$ has the reflection symmetry that $\forall A \in \mathcal{M}$, $-A \in \mathcal{M}$ and $m(-A) = m(A)$

1st proof: Prove the statement directly by the definition of $m$ (and $m^*$)

Key points: Given $A \subseteq \mathbb{R}$. $\{I_n : n \geq 1\}$ is an open-interval covering of $A$ if and only if $\{cI_n : n \geq 1\}$ is an open-interval covering of $cA$,

and $\ell(cI_n) = |c| \cdot \ell(I_n)$. $\Rightarrow$ $m^*(cA) = |c| \cdot m^*(A)$

Next, assume $A \in \mathcal{M}$. Argue that $\forall B \subseteq \mathbb{R}$. $m^*(B) = m^*(B \cap (cA)) + m^*(B \cap (cA)^c) \Rightarrow cA \in \mathcal{M}$

Treat the cases "$c=0$" and "$c \neq 0$" separately.

2nd proof: Prove the statement via $m$ restricted on $\mathcal{B}_{\mathbb{R}}$ (but a necessary element will be introduced in the next section)

Fix $c \in \mathbb{R} \setminus \{0\}$ ("$c=0$" case is trivial). Consider a set function $\mu_c$ on $\mathcal{B}_{\mathbb{R}}$ s.t.

$\forall B \in \mathcal{B}_{\mathbb{R}}$. $\mu_c(B) = m(cB)$. $\mu_c$ is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

It's clear that $\forall a, b \in \mathbb{R}$. $a<b$. $\forall n \geq 1$.

$$\mu_c((a,b) \cap [-n,n]) = |c|\, \ell((a,b) \cap [-n,n]) = |c| \cdot m((a,b) \cap [-n,n])$$

$\Rightarrow \mu_c = |c| \cdot m$ on $\mathcal{B}_{[-n,n]}$ $\Rightarrow$ $\mu_c = |c| \cdot m$ on $\mathcal{B}_{\mathbb{R}}$ $\Rightarrow \forall B \subseteq \mathcal{B}_{\mathbb{R}}$. $m(cB) = |c| \cdot m(B)$

However, we still need to show $\mu_c = |c| \cdot m$ on $\mathcal{M}$. We need to better understand the relation between $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{M}$

## §1.5. Relation between $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{M}$

We have proven that "$m$ is complete", i.e. $\forall A \in \mathcal{M}$ with $m(A) = 0$
$B \subseteq A \Rightarrow B \in \mathcal{M}$ and $m(B) = 0$

Now we introduce the notion of "completion" in the general setting.

Definition: Given a measure space $(X, \mathcal{F}, \mu)$, consider the collection of subsets of $X$
$\mathcal{N} := \{B \subseteq X : \exists A \in \mathcal{F}$ with $\mu(A) = 0$ s.t. $B \subseteq A\}$ (all subsets of null sets)

Then, $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N}) = \sigma(\{B \subseteq X : B \in \mathcal{F}$ or $B \in \mathcal{N}\})$ is called the completion of $\mathcal{F}$ with respect to $\mu$.

Proposition $\overline{\mathcal{F}} = \{F \subseteq X : \exists E, G \in \mathcal{F}$. s.t. $E \subseteq F \subseteq G$ and $\mu(G \setminus E) = 0\}$

Proof: Denote by $\mathcal{G}$ the collection on the RHS above. First, verify that $\mathcal{G}$ is a $\sigma$-algebra of subsets of $X$. Next, since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{N} \subseteq \mathcal{G}$,
$\overline{\mathcal{F}} = \sigma(\mathcal{F} \cup \mathcal{N}) \subseteq \mathcal{G}$. Meanwhile, for any $F \in \mathcal{G}$, $\exists E, G \in \mathcal{F}$ s.t.

$E \subseteq F \subseteq G$ and $\mu(G \setminus E) = 0$. $\Rightarrow F = E \cup (F \setminus E)$ where $F \setminus E \in \mathcal{N}$. $\Rightarrow F \in \overline{\mathcal{F}}$

Therefore $\mathcal{G} \subseteq \overline{\mathcal{F}}$. We conclude that $\mathcal{G} = \overline{\mathcal{F}}$

Definition Given measure space $(X, \mathcal{F}, \mu)$, $\mu$ can be extended the $\overline{\mathcal{F}}$ as

$\forall F \in \overline{\mathcal{F}}$. if $E \subseteq F \subseteq G$ for some $E, G \in \mathcal{F}$ with $\mu(G \setminus E) = 0$, then $\mu(F) := \mu(E) = \mu(G)$.

(Verify that $\mu$ is well-defined on $F$, i.e. if $\exists E', G' \in \mathcal{F}$ s.t. $E' \subseteq F \subseteq G'$ with $\mu(G' \setminus E') = 0$, then it must be that $\mu(E) = \mu(E') = \mu(G) = \mu(G')$ )

$\mu : \overline{\mathcal{F}} \to [0, \infty]$ is again a measure. $(X, \overline{\mathcal{F}}, \mu)$ is the completion of $(X, \mathcal{F}, \mu)$

$(X, \overline{\mathcal{F}}, \mu)$ is a complete measure space in the sense that $\forall A \subseteq X$.

if $\exists B \in \overline{\mathcal{F}}$ with $\mu(B) = 0$ s.t. $A \subseteq B$. then $A \in \overline{\mathcal{F}}$ and $\mu(A) = 0$.

Theorem $(\mathbb{R}, \mathcal{M}, m)$ is the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$. That is, $\forall A \in \mathcal{M}$, $\exists B, C \in \mathcal{B}_{\mathbb{R}}$ s.t. $B \subseteq A \subseteq C$ and $m(C \setminus B) = 0$. (Every Lebesgue measurable set differs from a Borel set by at most a null set)

Proof: By the results (1) (2) from the regularity theorem of $m$.

$\forall n \geq 1$, $\exists$ open set $G_n$ and closed set $F_n$. s.t. $F_n \subseteq A \subseteq G_n$ and $m(G_n \setminus A) \leq \frac{1}{n}$. $m(A \setminus F_n) \leq \frac{1}{n}$.

Set $C = \bigcap_{n=1}^{\infty} G_n$. $B = \bigcup_{n=1}^{\infty} F_n$. Clearly, $B, C \in \mathcal{B}_{\mathbb{R}}$ and $B \subseteq A \subseteq C$

Moreover, $m(A \setminus B) \leq m(A \setminus F_n) \leq \frac{1}{n}$ and $m(C \setminus A) \leq m(G_n \setminus A) \leq \frac{1}{n}$ $\forall n \geq 1$

$\Rightarrow$ $m(C \setminus B) = m(A \setminus B) + m(C \setminus A) \leq \frac{2}{n}$ $\forall n \geq 1$. $\Rightarrow$ $m(C \setminus B) = 0$ $\square$

Now going back to the 2nd proof of the proposition on the rescaling property of $m$ ...

We already know that. $\forall c \in \mathbb{R}$, $\forall B \in \mathcal{B}_{\mathbb{R}}$, $m(cB) = |c| \cdot m(B)$. Then, it follows immediately from the theorem above that $\forall c \in \mathbb{R}$. $\forall A \in \mathcal{M}$. $m(cA) = |c| \cdot m(A)$.

## §1.6. Some special sets.

We want to answer the following questions...

Q1. Is there $A \in \mathcal{M}$ with $m(A) = 0$ but $A$ is uncountable?

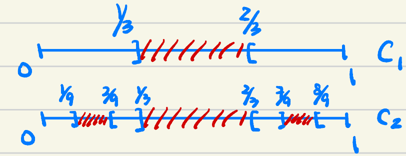
Q2: Is there $A \subseteq \mathbb{R}$ that $A \notin \mathcal{M}$? (If so, are they rare or abundant?)

Q3: Is there $A \in \mathcal{M}$ but $A \notin \mathcal{B}_{\mathbb{R}}$?

Yes to all.

Q1 : There exists $A \in \mathcal{M}$ with $m(A) = 0$ and $A$ is uncountable

A classical example is the Cantor set




$C_0 = [0,1] \xrightarrow[\text{removing mid } 1/3]{} C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$\xrightarrow[\text{removing mid } 1/3 \text{ from each segment}]{} C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

... repeat this process.

$\forall n \geq 1$. $C_n$ is the union of $2^n$ disjoint closed intervals each of which has length $\frac{1}{3^n}$. and $C_n \downarrow$

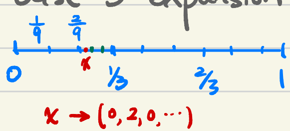
The Cantor set is $C := \bigcap_{n=0}^{\infty} C_n$. Then, (1) $C$ is closed . and hence $C \in \mathcal{B}_{\mathbb{R}}$.

(2) $\forall n \geq 1$, $m(C_n) = 2^n \cdot \frac{1}{3^n} \Rightarrow m(C) = \lim_{n \to \infty} m(C_n) = 0$.

(3) $C$ is uncountable (in fact, $C$ has the same cardinality as $[0,1]$).

To see (3). consider the base-3 expansion of numbers in $[0,1]$ : $\forall x \in [0,1]$, $\exists \{a_n : n \geq 1\} \in \{0,1,2\}^{\mathbb{N}}$

s.t. $x = \sum_{n=1}^{\infty} a_n \cdot \frac{1}{3^n}$




Some values have two expansions, e.g. $\frac{1}{3} \nearrow (1,0,0,\cdots) \searrow (0,2,2,\cdots)$ , $\frac{2}{3} \nearrow (1,2,2,2,\cdots) \searrow (2,0,0,\cdots)$

Then, $C = \{x \in [0,1] : x$ admits a base-3 expansion $(a_1, a_2, \cdots a_n, \cdots)$ where $a_n \in \{0,2\} \ \forall n \geq 1\}$

Note that the set on the RHS does contain values such as $\frac{1}{3}$, $\frac{2}{3}$. $\frac{1}{9}$. $\frac{2}{9}$, ...

Define $f: C \to [0,1]$: $\forall x \in C$. assume the base-3 expansion of $x$ is $(a_1, a_2, a_3, \cdots)$. $a_n \in \{0,2\}$ $\forall n \geq 1$.

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2} \frac{1}{2^n}$$ (that is, replace the 2s by 1s, and turn base-3 to base-2)

Observe that $f$ is a surjection. $\forall y \in [0,1]$, $\exists$ $(b_1, b_2, \cdots)$ s.t. $b_n \in \{0,1\}$ $\forall n \geq 1$

s.t. $y = \sum_{n=1}^{\infty} b_n \frac{1}{2^n}$ (binary expansion of $y$)

$\Rightarrow$ $y = f(x)$ for $x \in C$ with base-3 expansion $(2b_1, 2b_2, 2b_3 \cdots)$

i.e. $x = \sum_{n=1}^{\infty} (2b_n) \cdot \frac{1}{3^n}$

$\Rightarrow$ $C$ has cardinality no smaller than $[0,1]$, so $C$ is uncountable.

(Since $C \subseteq [0,1]$, $C$ has the same cardinality as $[0,1]$)

<u>Remark</u>: One can similarly construct Cantor-like set under different bases.