Sep. 6

# Lecture 4

(8) If $A = \bigcup_{k=1}^{\infty} J_k$ where $J_k$'s are almost disjoint intervals (at most sharing end points)

then $m^*(A) = \sum_{k=1}^{\infty} \ell(J_k)$

Proof: W.L.O.G. assume $\ell(J_k) < \infty$ $\forall\, k \geq 1$. Fix an arbitrary $\varepsilon > 0$.

for each $k \geq 1$, choose open interval $I_k \subset J_k$ s.t. $\ell(J_k) \leq \ell(I_k) + \varepsilon/2^k$

For each integer $N > 1$, $I_1, \cdots, I_N$ are disjoint with a positive distance from one another. By (7) (induction to N times),

$$m^*\left(\bigcup_{k=1}^{N} I_k\right) = \sum_{k=1}^{N} \ell(I_k) \geq \sum_{k=1}^{N} \ell(J_k) - \sum_{k=1}^{N} \frac{\varepsilon}{2^k} \geq \sum_{k=1}^{N} \ell(J_k) - \varepsilon$$

Since $\bigcup_{k=1}^{N} I_k \subseteq A$, $m^*(A) \geq \sum_{k=1}^{N} \ell(J_k) - \varepsilon \overset{\text{as } N\to\infty}{\Longrightarrow} m^*(A) \geq \sum_{k=1}^{\infty} \ell(J_k) - \varepsilon$.

On the other hand, $m^*(A) \leq \sum_{k=1}^{+\infty} \ell(J_k)$ by (countable subadditivity) $\square$.

Despite of all the properties above, $m^*$ doesn't support "$m^*(A \cup B) = m^*(A) + m^*(B)$" for arbitrary disjoint sets A, B. Consider restricting $m^*$ to the "good" sets.

## Step 2 : Define measurable sets

**Definition** A set $A \subseteq \mathbb{R}$ is $m^*$-measurable if $\forall B \subseteq \mathbb{R}$

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c)$$

Otherwise, $A$ is a non-measurable set.

**Remark**: By subaddtivity, we know that $\forall A, B \subseteq \mathbb{R}$

$$m^*(B) \leq m^*(B \cap A) + m^*(B \cap A^c)$$

So, $m^*$-measurability of $A$ is about whether or not "$<$" could occur.

**Theorem** (Carathéodary's Theorem) Let $\mathcal{M} := \{A \subseteq \mathbb{R} : A \text{ is } m^*\text{-measurable}\}$

Then, $\mathcal{M}$ is a $\sigma$-algebra (of subsets of $\mathbb{R}$). Define $m : \mathcal{M} \to [0, \infty]$

by $\forall A \in \mathcal{M}$ $\quad m(A) = m^*(A)$. Then, $m$ is a measure on $(\mathbb{R}, \mathcal{M})$

$m$ is called the Lebesgue measure, and $A \in \mathcal{M}$ is a (Lebesgue) measurable set.

Proof: It follows immediately from the definition of $m^*$-measurability that $\mathbb{R} \in \mathcal{M}$, and if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$.

Next, we will show $\mathcal{M}$ is closed under finite union. i.e. if $A_1, \cdots, A_N \in \mathcal{M}$, then $\bigcup_{n=1}^{N} A_n \in \mathcal{M}$. It's sufficient to treat the case $N=2$.

Given $A_1, A_2 \in \mathcal{M}$. $\forall B \subseteq \mathbb{R}$

$$\begin{aligned} m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\ &= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap \underbrace{A_1^c \cap A_2^c}_{B \cap (A_1 \cup A_2)^c}) \\ &\overset{\text{subadditivity}}{\geq} m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c) \end{aligned}$$

union $= B \cap (A_1 \cup A_2)$

Subadditivity implies the reverse ineq. So, we have proven that

$$\forall B \subseteq \mathbb{R}. \quad m^*(B) = m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c) \Rightarrow A_1 \cup A_2 \in \mathcal{M}.$$

Now, consider a sequence $\{A_n : n \geq 1\} \subseteq \mathcal{M}$. We want to show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

W.L.O.G., we may assume $A_n$'s are disjoint. (Otherwise, we replace $\{A_n : n \geq 1\}$ by $\{B_n : n \geq 1\}$ where $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$. Then, $\{B_n : n \geq 1\} \subseteq \mathcal{M}$ (since we have shown $\mathcal{M}$ is closed under finite union and complement). $B_n$'s are disjoint.

and $\bigsqcup_{n=1}^{\infty} B_n = \bigsqcup_{n=1}^{\infty} A_n$. So, it's equivalent to show $\bigsqcup_{n=1}^{\infty} B_n \in \mathcal{M}$.)

For every $n \geq 1$, set $E_n := \bigsqcup_{i=1}^{n} A_i$. We already know that $E_n \in \mathcal{M}$ $\forall n \geq 1$.

$\forall B \subseteq \mathbb{R}$,

$$
\begin{aligned}
m^*(B) &= m^*(B \cap E_n) + m^*(B \cap E_n^c) \qquad \text{because } E_n \subseteq \bigsqcup_{i=1}^{\infty} A_i \\
&\geq m^*(B \cap E_n) + m^*\Big(B \cap \Big(\bigsqcup_{i=1}^{\infty} A_i\Big)^c\Big) \\
&\qquad\quad \cap A_n \quad\quad \cap A_n^c \\
&= m^*(B \cap E_n \cap A_n) + m^*(B \cap E_n \cap A_n^c) + m^*\Big(B \cap \Big(\bigsqcup_{i=1}^{\infty} A_i\Big)^c\Big) \\
&\qquad\quad \overset{\shortparallel}{B \cap A_n} \qquad\qquad\qquad \overset{\shortparallel}{B \cap E_{n-1}} \\
&= m^*(B \cap A_n) + m^*(B \cap E_{n-1}) + m^*\Big(B \cap \Big(\bigsqcup_{i=1}^{\infty} A_i\Big)^c\Big) \\
&\qquad\qquad\qquad\qquad \cap A_{n-1} \quad \cap A_{n-1}^c \\
&= m^*(B \cap A_n) + m^*(B \cap A_{n-1}) + m^*(B \cap E_{n-2}) + m^*\Big(B \cap \Big(\bigsqcup_{i=1}^{\infty} A_i\Big)^c\Big) \\
&\;\;\vdots \\
&= \sum_{i=1}^{n} m^*(B \cap A_i) + m^*\Big(B \cap \Big(\bigsqcup_{i=1}^{\infty} A_i\Big)^c\Big)
\end{aligned}
$$

Since $n$ is arbitrary, $m^*(B) \geq \sum_{n=1}^{\infty} m^*(B \cap A_n) + m^*\Big(B \cap \Big(\bigsqcup_{n=1}^{\infty} A_n\Big)^c\Big)$

(Countable subadd.) $\geq m^*\Big(B \cap \Big(\bigsqcup_{n=1}^{\infty} A_n\Big)\Big) + m^*\Big(B \cap \Big(\bigsqcup_{n=1}^{\infty} A_n\Big)^c\Big)$

Thus, $\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{M}$. We have proven that $\mathcal{M}$ is a $\sigma$-algebra.

Now, we move onto proving $m = m^*|_{\mathcal{M}}$ is a measure. $m(\phi) = m^*(\phi) = 0$ obviously.

Assume $\{A_n : n \geq 1\} \subseteq \mathcal{M}$ is a sequence of disjoint measurable sets. First, by countable subadd. (of $m^*$)

$$m\left(\bigsqcup_{n=1}^{\infty} A_n\right) = m^*\left(\bigsqcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n) = \sum_{n=1}^{\infty} m(A_n)$$

Second, by monotonicity of $m^*$, $\forall n \geq 1$. $m\left(\bigsqcup_{n=1}^{\infty} A_n\right) = m^*\left(\bigsqcup_{n=1}^{\infty} A_n\right) \geq m^*\left(\bigsqcup_{i=1}^{n} A_i\right) = m\left(\bigsqcup_{i=1}^{n} A_i\right)$

Since $A_1, \cdots, A_n \in \mathcal{M}$ are disjoint, we can follow a similar argument as above to prove

$$m^*\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i), \quad \text{or equivalently} \quad m\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i)$$

Therefore. $\forall n \geq 1$. $m\left(\bigsqcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{n} m(A_i) \xrightarrow{\text{as } n \to \infty} \sum_{i=1}^{\infty} m(A_i)$

We have proven $m\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ (countable add.) $\Rightarrow m$ is a measure.

<u>Proposition</u> $\mathcal{M}$ and $m$ are translation invariant, i.e., $\forall A \in \mathcal{M}$, $\forall x \in \mathbb{R}$, $A + x \in \mathcal{M}$ and $m(A) = m(A+x)$

Proof: Given $A \in \mathcal{M}$ and $x \in \mathbb{R}$. $\forall B \subseteq \mathbb{R}$. $m^*(B) = m^*(B-x) = m^*((B-x) \cap A) + m^*((B-x) \cap A^c)$

translation invariance of $m^*$

$$= m^*(B \cap (A+x)) + m^*(B \cap (A+x)^c)$$

Therefore, $A + x \in \mathcal{M}$. Moreover, $m(A) = m^*(A) = m^*(A+x) + m(A+x)$.

**Theorem.** $\forall a, b \in \mathbb{R}$ $a < b$. $(a, b) \in \mathcal{M}$ and $m((a,b)) = b - a$.

**Important corollary:** $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$. i.e. all Borel sets are Lebesgue measurable