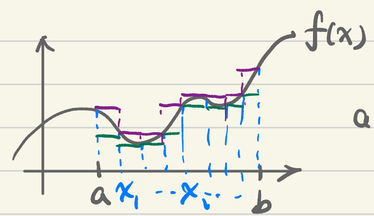
Aug. 28 Lecture 1.

- Discuss course outline. course logistics. calendar. etc.

Motivation of Lebesgue integration:

Review of Riemann integral

"$\int_a^b f(x)\,dx$" = area of the region under graph of f



$a = x_0 < x_1 < \cdots < x_n = b$ partition of $[a,b]$

$\Delta x_i = x_i - x_{i-1}$, $i = 1, \cdots, n$

Upper integral $\overline{\int_a^b} f(x)\,dx = \inf \left\{ \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f \cdot \Delta x_i \; : \; a = x_0 < x_1 < \cdots < x_n = b \right\}$

limit as $\max_i |\Delta x_i| \to 0$

Lower integral $\underline{\int_a^b} f(x)\,dx = \sup \left\{ \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f \cdot \Delta x_i \; : \; a = x_0 < x_1 < \cdots < x_n = b \right\}$

f is Riemann integrable if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx =: \int_a^b f(x)\,dx \to$ Riemann integral of f over $[a,b]$

Recall that f is Riemann integrable if (i) either f is continuous on $[a,b]$;

(ii) or f is monotone on $[a,b]$; (iii) or f is bounded on $[a,b]$, continuous except at possibly finitely many points.

However, many functions are NOT Riemann integrable

Example: $f(x)$ on $[a,b]$ s.t. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [a,b] \end{cases}$

Since both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense on $\mathbb{R}$.

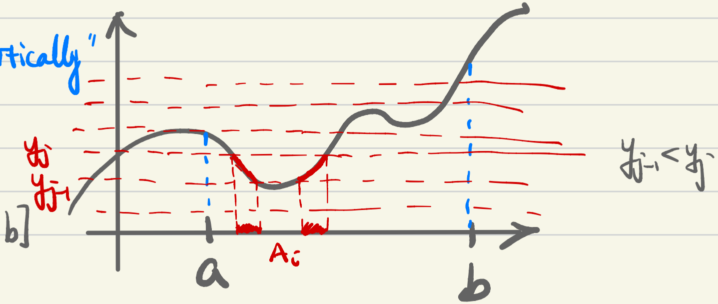
$\forall\ a = x_0 < x_1 < \cdots < x_n = b \quad \forall\ i = 1, \cdots, n. \quad \sup_{[x_{i-1}, x_i]} f = 1$ and $\inf_{[x_{i-1}, x_i]} f = 0$

$\Rightarrow 1 = \overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx = 0 \Rightarrow f$ is NOT Riemann integrable on $[a,b]$.

We need to a more general notion of "integral" that applies to more general functions.

Instead of dividing the region under graph of f "vertically"

Consider slicing it "horizontally"



Each slice corresponds to a subset of $[a,b]$

$A_j := \{x \in [a,b] : y_{j-1} < f(x) \le y_j\}$

Then, the contribution of this slice to the area is "approximately"

$$y_{j-1} \cdot \text{"}\underbrace{\text{measure}}_{\text{size}} \text{ of } A_j\text{"}$$

$$\Rightarrow \text{total area} \approx \sum_j y_{j-1} \cdot \text{"measure of } A_j\text{"}$$ (gist of Lebesgue integral)

In order to carry out such an idea, we need a more general notion of "measure" for general sets (e.g. A_j may not be interval, not even union of intervals)

This is the motivation of Lebesgue measure

Chapter 1 Measures

For now, what we have in mind is "measure on $\mathbb{R}$", but if possible, we will present statements in the general setting.

"Measure is a general and abstract concept. $\mathbb{R}/\mathbb{R}^n$ is just a particular example of measure space.

First, we need to identify the collection of sets that we want to "measure"

§1.1 σ- algebras

We can't measure the size of an arbitrary set. We need some restrictions for "measurable sets"

Definition: Let X be a space (i.e., a non-empty set; e.g., X can be $\mathbb{R}$, $\mathbb{R}^n$, subset of $\mathbb{R}$, $\cdots$)
$\mathcal{F}$ is a collection of subsets of X. $\mathcal{F}$ is called a σ-algebra ("sigma-algebra" or σ-field) of subsets of X. if

(1) $X \in \mathcal{F}$.

(2) if $A \in \mathcal{F}$. then $A^c := X \setminus A \in \mathcal{F}$ (closed under taking complement)

(3) if $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under taking countable union)

Note !! The elements in $\mathcal{F}$ are SUBSETS of X

e.g., If $X = \mathbb{R}$, then $\mathcal{F}$ may contain $\mathbb{R}$, (a, b), $\mathbb{Q}$. $(-\infty, a)$, $\{c\}$ $(c \in \mathbb{R})$ $\cancel{c \in \mathcal{F}}$.

Definition of σ- algebra $\mathcal{F}$ leads to

(4) $\phi \in \mathcal{F}$ (because $\phi = X^c$ and $X \in \mathcal{F}$)

(5) if $\{A_n : n \geq 1\} \subseteq \mathcal{F}$. then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$. (closed under countable intersection)

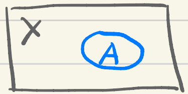
(6) if $A_1, \cdots A_N \in \mathcal{F}$, then $\bigcup_{n=1}^{N} A_n \in \mathcal{F}$ and $\bigcap_{n=1}^{N} A_n \in \mathcal{F}$ (closed under finite union/intersection)

(7) if $A, B \in \mathcal{F}$. then $B \setminus A$, $A \setminus B$, $A \triangle B \in \mathcal{F}$ (closed under taking difference)

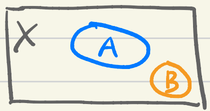
$(B \setminus A) \overset{"}{\sqcup} (A \setminus B)$

Purpose of σ-algebra :
- collection of "measurable sets" is a σ-algebra
- "measure" is defined on a σ-algebra

Examples of σ-algebras :
- "smallest" $\{\phi, X\}$, "largest" $2^X := \{\text{all subsets of } X\}$



- if $A \subseteq X$. then $\{\phi, X, A, A^c\}$ is a σ-algebra



- if $A, B \subseteq X$, $A \cap B = \phi$, then

  $\{\phi, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$ is a σ-algebra.

The examples above suggest that :

① possible to compare σ-algebras : $\mathcal{F}_1, \mathcal{F}_2$ two σ-algebras of subsets of X.

$\mathcal{F}_1$ is smaller than $\mathcal{F}_2$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2$

② some σ-algebras can be generated by a collection of subsets of X

Definition: Let $\mathcal{C}$ be a collection of subsets of X. Then, the σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, is a σ-algebra of subsets of X s.t. ① $\mathcal{C} \subseteq \sigma(\mathcal{C})$

and ② if $\mathcal{F}$ is another σ-algebra s.t. $\mathcal{C} \subseteq \mathcal{F}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

That is, $\sigma(\mathcal{C})$ is the smallest σ-algebra that is a superset of $\mathcal{C}$.

If $\mathcal{F} = \sigma(\mathcal{C})$, then $\mathcal{C}$ is called an generator of $\mathcal{F}$.

Proposition 1. Given $\mathcal{C}$ a collection of subsets of X, $\sigma(\mathcal{C}) = \bigcap_{\substack{\{\mathcal{F}:\ \sigma\text{-algebra} \\ \text{s.t. } \mathcal{C} \subseteq \mathcal{F}\}}} \mathcal{F}$

2. If $\mathcal{C}$ is a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$.

3. Given $\mathcal{C}_1, \mathcal{C}_2$ two collections of subsets of X. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$.